

Generalized Ulam-Hyers Stability of a Fractional System of Nonlinear Integro-Differential Equations

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Abstract

In this paper, we consider a fractional system of nonlinear integro-differential equations involving $(n^2 + n)$ -Caputo derivatives. We discuss the existence and uniqueness of solution and the existence of at least one solution, using some fixed point theorems. Moreover, we present some types of Ulam stability results for such system. In addition, some examples are provided to illustrate the applications of our main results.

Keywords: Caputo derivative, fixed point, fractional integro-differential equation, existence and uniqueness, generalized Ulam-Hyers stability.

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1 Introduction and Preliminaries

The fractional calculus represents a powerful tool for studying many problems in applied mathematics and engineering sciences. For more details, we refer the reader to [10, 12, 16, 17, 18] and the reference therein. Moreover, the study of fractional differential equations is also important in various problems of applied sciences, and has attracted the attention of many authors, see [1, 2, 3, 4, 11, 13, 14, 15, 25, 26]. On the other hand, the study of Ulam type stability problems has grown to be one of the most important subjects in mathematical analysis. In [5, 6, 7, 8, 9, 19, 20, 21, 22, 23], Z. Dahmani and A. Taïeb have obtained new results on the existence and uniqueness of solutions in addition to some types of Ulam stability for fractional differential equations.

Now, let us present some results that motivated this paper: In [27], J. Wang, L. Lv and Y. Zhou studied different types of Ulam stability for the following fractional differential equation:

$${}^c D^\alpha x(t) = f(t, x(t)), \quad t \in [a, b],$$

where ${}^c D^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$ and the function f satisfies some specific conditions.

In [15], R.W. Ibrahim investigated the existence and uniqueness of solutions in addition to different types of Ulam-Hyers stability for the following problem:

$$\begin{cases} D_z^\alpha u(z) = f(z, u(z)), \\ au(0) + bu(1) = cu(\xi), \quad z, \xi \in U, \quad a + b \neq c, \end{cases}$$

where $u : U \rightarrow B$ is an analytic function for all z in the unit disk $U := \{z : |z| < 1\}$ and $f : U \times B \rightarrow B$ is an analytic function in $z \in U$. B is the space of all analytic and bounded functions in the unit disk.

In [24], A. Taïeb discussed the existence, uniqueness and the Ulam-Hyers stability for the following system:

$$\begin{cases} D^{\alpha_1} u_1(t) = f_1 \left(t, u_1(t), \dots, u_n(t), D^{\alpha_1^1} u_1(t), \dots, D^{\alpha_1^{n-1}} u_1(t), \right. \\ \quad \left. D^{\alpha_2^1} u_2(t), \dots, D^{\alpha_2^{n-1}} u_2(t), \dots, \right. \\ \quad \left. D^{\alpha_n^1} u_n(t), \dots, D^{\alpha_n^{n-1}} u_n(t) \right), \\ \vdots \\ D^{\alpha_n} u_n(t) = f_n \left(t, u_1(t), \dots, u_n(t), D^{\alpha_1^1} u_1(t), \dots, D^{\alpha_1^{n-1}} u_1(t), \right. \\ \quad \left. D^{\alpha_2^1} u_2(t), \dots, D^{\alpha_2^{n-1}} u_2(t), \dots, \right. \\ \quad \left. D^{\alpha_n^1} u_n(t), \dots, D^{\alpha_n^{n-1}} u_n(t) \right), \\ 0 < t \leq 1, \quad n-1 < \alpha_k < n, \quad k = 1, 2, \dots, n, \quad i-1 < \alpha_k^i < i, \quad i = 1, 2, \dots, n-1, \\ u_k^{(j)}(0) = \omega_j^k, \quad D^{\nu_k} u_k(1) + J^{\eta_k} u_k(1) = 0, \quad n-2 < \nu_k < n-1, \quad \eta_k > 0, \\ n \in \mathbb{N} - \{0, 1\}, \end{cases}$$

where $f_k : (0, 1] \times \mathbb{R}^{n^2} \rightarrow \mathbb{R}$ are continuous functions, singular at $t = 0$, and $\lim_{t \rightarrow 0^+} f_k(t) = \infty$. D^{α_k} and $D^{\alpha_k^i}$, $i = 1, 2, \dots, n-1$, $k = 1, 2, \dots, n$, are in the sense of Caputo.

In this paper, we present new results on the existence, uniqueness and some types of Ulam stability for the following fractional system of nonlinear

integro-differential equations:

$$\left\{ \begin{array}{l} D^{\alpha_1} u_1(t) = f_1 \left(\begin{array}{l} t, u_1(t), \dots, u_n(t), D^{a_1^1} u_1(t), \dots, D^{a_1^{n-1}} u_1(t), \\ D^{a_2^1} u_2(t), \dots, D^{a_2^{n-1}} u_2(t), \dots, \\ D^{a_n^1} u_n(t), \dots, D^{a_n^{n-1}} u_n(t), \\ \int_0^t h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^t h_n(\tau, u_n(\tau)) d\tau \end{array} \right), \\ \vdots \\ D^{\alpha_n} u_n(t) = f_n \left(\begin{array}{l} t, u_1(t), \dots, u_n(t), D^{a_1^1} u_1(t), \dots, D^{a_1^{n-1}} u_1(t), \\ D^{a_2^1} u_2(t), \dots, D^{a_2^{n-1}} u_2(t), \dots, \\ D^{a_n^1} u_n(t), \dots, D^{a_n^{n-1}} u_n(t), \\ \int_0^t h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^t h_n(\tau, u_n(\tau)) d\tau \end{array} \right), \\ t \in J, \\ n-1 < \alpha_k < n, \quad k = 1, 2, \dots, n, \quad i-1 < \alpha_k^i < i, \quad i = 1, 2, \dots, n-1, \\ u_k^{(j)}(0) = a_j^k, \quad j = 0, 1, \dots, n-2, \\ u_k^{(n-1)}(1) = D^{\mu_k} u_k(1), \quad n-2 < \mu_k < n-1, \quad k = 1, 2, \dots, n, \\ n \in \mathbb{N} - \{0, 1\}, \end{array} \right. \quad (1)$$

where $J := [0, 1]$, $f_k : [0, 1] \times \mathbb{R}^{n^2+n} \rightarrow \mathbb{R}$ are continuous functions, D^{α_k} and $D^{\alpha_k^i}$, $i = 1, 2, \dots, n-1$, $k = 1, 2, \dots, n$, denote the Caputo fractional derivatives.

To the best of our knowledge, there are no papers that have considered this kind of fractional coupled system.

Let us recall some basic definitions and facts from the fractional calculus which can be found in [16].

Definition 1.1 *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[0, \infty)$ is defined as:*

$$J^\varphi f(t) = \begin{cases} \frac{1}{\Gamma(\varphi)} \int_0^t (t-s)^{\varphi-1} f(s) ds, & \varphi > 0, \\ f(t), & \varphi = 0, \end{cases}$$

where $t \geq 0$, and $\Gamma(\varphi) := \int_0^{+\infty} e^{-x} x^{\varphi-1} dx$.

Definition 1.2 *The Caputo derivative of order α for a function $u : [0, \infty) \rightarrow \mathbb{R}$, which is at least n -times differentiable can be defined as:*

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds = J^{n-\alpha} u^{(n)}(t), \quad \alpha > 0,$$

for $n-1 < \alpha < n$, $n \in \mathbb{N} - \{0\}$.

Lemma 1.3 Let $n \in \mathbb{N} - \{0\}$, $n - 1 < \alpha < n$, and $D^\alpha u(t) = 0$. Then, $u(t) = \sum_{j=0}^{n-1} c_j t^j$, and $J^\alpha D^\alpha u(t) = u(t) + \sum_{j=0}^{n-1} c_j t^j$, $(c_j)_{j=0,1,\dots,n-1} \in \mathbb{R}$.

Lemma 1.4 (Schaefer Fixed Point Theorem) Let E be a Banach space and assume that $A : E \rightarrow E$ is completely continuous operator. If the set $V := \{x \in E : x = \mu Ax, 0 < \mu < 1\}$ is bounded, then A has a fixed point in E .

Let us now impart the integral solution of system (1) by the following auxiliary result:

Lemma 1.5 Let $n \in \mathbb{N} - \{0, 1\}$, $n - 1 < \alpha_k < n$, $k = 1, 2, \dots, n$, and $A_k \in C(J, \mathbb{R})$. Then, the unique solution of the following problem

$$\begin{cases} D^{\alpha_1} u_1(t) = A_1(t), \quad t \in J, \\ \vdots \\ D^{\alpha_n} u_n(t) = A_n(t), \quad t \in J, \\ u_k^{(j)}(0) = a_j^k, \quad j = 0, 1, \dots, n-2, \\ u_k^{(n-1)}(1) = D^{\mu_k} u_k(1), \quad n-2 < \mu_k < n-1, \end{cases}$$

is given by $(u_1, u_2, \dots, u_n)(t)$;

$$\begin{aligned} u_k(t) = & \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} A_k(s) ds + \sum_{j=0}^{n-2} \frac{a_j^k}{j!} t^j \\ & + \frac{\Gamma(n-\mu_k) t^{n-1}}{(\Gamma(n-\mu_k)-1)\Gamma(n)} \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} A_k(s) ds, \end{aligned} \quad (2)$$

where $k = 1, 2, \dots, n$.

Proof. Thanks to Lemma 1.3, we get

$$\begin{cases} u_1(t) = \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} A_1(s) ds - \sum_{j=0}^{n-1} c_j^1 t^j, \\ \vdots \\ u_n(t) = \int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} A_n(s) ds - \sum_{j=0}^{n-1} c_j^n t^j, \end{cases} \quad (3)$$

such that

$$\begin{pmatrix} c_0^1 & c_1^1 & \dots & c_{n-1}^1 \\ c_0^2 & c_1^2 & \dots & c_{n-1}^2 \\ \dots & \dots & \dots & \dots \\ c_0^n & c_1^n & \dots & c_{n-1}^n \end{pmatrix} \in M_n(\mathbb{R}).$$

For all $k = 1, \dots, n$, we observe that

$$\begin{cases} u_k^{(j)}(0) = -j!c_j^k, & j = 0, 1, \dots, n-2, \\ u_k^{(n-1)}(1) = -(n-1)!c_{n-1}^k, \\ D^{\mu_k}u_k(1) = \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} A_k(s) ds - \frac{\Gamma(n)}{\Gamma(n-\mu_k)} c_{n-1}^k. \end{cases}$$

By the conditions: $u_k^{(j)}(0) = a_j^k$, $j = 0, 1, \dots, n-2$, and $u_k^{(n-1)}(1) = D^{\mu_k}u_k(1)$, we obtain

$$c_j^k = \begin{cases} -\frac{a_j^k}{j!}, & j = 0, 1, \dots, n-2, \\ \frac{\Gamma(n-\mu_k)}{\Gamma(n)(1-\Gamma(n-\mu_k))} \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} A_k(s) ds, & j = n-1. \end{cases} \quad (4)$$

Substituting (4) in (3), we have (2). This ends the proof.

We introduce the Banach space

$$B := \left\{ (u_1, u_2, \dots, u_n) : u_k \in C([0, 1], \mathbb{R}), D^{\alpha_k^i}u_k \in C([0, 1], \mathbb{R}), \right. \\ \left. k = 1, 2, \dots, n, i = 1, 2, \dots, n-1 \right\};$$

endowed with the norm:

$$\|(u_1, u_2, \dots, u_n)\|_B = \max_{\substack{1 \leq k \leq n \\ 1 \leq i \leq n-1}} \left(\|u_k\|_\infty, \|D^{\alpha_k^i}u_k\|_\infty \right),$$

where, $n \in \mathbb{N} - \{0, 1\}$, $\|u_k\|_\infty = \max_{t \in J} |u_k(t)|$, $\|D^{\alpha_k^i}u\|_\infty = \max_{t \in J} |D^{\alpha_k^i}u_k(t)|$.

2 Existence of Solutions

Now, we are ready to establish sufficient conditions for the existence and the uniqueness of solutions for (1). Then, we present some examples to illustrate the applications of our Theorems.

Denote, for all $k = 1, \dots, n$, $i = 1, 2, \dots, n-1$, $n \in \mathbb{N} - \{0, 1\}$,

$$\begin{aligned} M_k &= \frac{1}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(n - \mu_k)}{|\Gamma(n - \mu_k) - 1| \Gamma(n) \Gamma(\alpha_k - \mu_k + 1)}, \\ \Delta_k &= \sum_{j=1}^{n^2} \lambda_j^k + \sum_{j=1}^n \lambda_{n^2+j}^k \omega_j, \\ M_k^i &= \frac{1}{\Gamma(\alpha_k - \alpha_k^i + 1)} + \frac{\Gamma(n - \mu_k)}{\Gamma(n - \alpha_k^i) |\Gamma(n - \mu_k) - 1| \Gamma(\alpha_k - \mu_k + 1)}. \end{aligned}$$

For the sake of convenience, we list the following hypotheses:

(H₁) : There exist nonnegative constants $(\lambda_j^k)_{j=1, \dots, n^2+n}^{k=1, \dots, n}$, and $(\omega_j)_{j=1, \dots, n}$, such that:

$$\forall t \in [0, 1], \forall (x_1, \dots, x_{n^2+n}), (y_1, \dots, y_{n^2+n}) \in \mathbb{R}^{n^2+n} :$$

$$|f_k(t, x_1, \dots, x_{n^2+n}) - f_k(t, y_1, \dots, y_{n^2+n})| \leq \sum_{j=1}^{n^2+n} \lambda_j^k |x_j - y_j|,$$

and

$$|h_j(t, x_j) - h_j(t, y_j)| \leq \omega_j |x_j - y_j|, \quad j = 1, \dots, n.$$

(H₂) :

$$\Theta := \max_{\substack{1 \leq k \leq n \\ 1 \leq i \leq n-1}} (M_k, M_k^i) \Delta_k < 1.$$

(H₃) : The functions

$$(f_k)_{k=1,2,\dots,n} : [0, 1] \times \mathbb{R}^{n^2+n} \rightarrow \mathbb{R}, \quad (h_k)_{k=1,2,\dots,n} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R},$$

are continuous.

Define the nonlinear operator $T : B \rightarrow B$ by

$$T(u_1, u_2, \dots, u_n)(t) := (T_1(u_1, u_2, \dots, u_n)(t), \dots, T_n(u_1, u_2, \dots, u_n)(t)),$$

where $t \in J$,

$$\begin{aligned} & T_k(u_1, u_2, \dots, u_n)(t) \\ : &= \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds \\ &+ \sum_{j=0}^{n-2} \frac{a_j^k}{j!} t^j + \frac{\Gamma(n - \mu_k) t^{n-1}}{(\Gamma(n - \mu_k) - 1) \Gamma(n)} \\ &\times \int_0^1 \frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds. \end{aligned} \quad (5)$$

Obviously, we have

$$\begin{aligned}
& D^{\alpha_k^i} T_k(u_1, u_2, \dots, u_n)(t) \\
: &= \int_0^t \frac{(t-s)^{\alpha_k - \alpha_k^i - 1}}{\Gamma(\alpha_k - \alpha_k^i)} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_s^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_s^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds \\
&+ \sum_{j=i}^{n-2} \frac{a_j^k}{\Gamma(j+1 - \alpha_k^i)} t^{j-\alpha_k^i} + \frac{\Gamma(n - \mu_k) t^{n-1-\alpha_k^i}}{\Gamma(n - \alpha_k^i) (\Gamma(n - \mu_k) - 1)} \\
&\times \int_0^1 \frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_s^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_s^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds, \quad (6)
\end{aligned}$$

where $i = 1, 2, \dots, n-2$,

$$\begin{aligned}
& D^{\alpha_k^{n-1}} T_k(u_1, u_2, \dots, u_n)(t) \\
: &= \int_0^t \frac{(t-s)^{\alpha_k - \alpha_k^{n-1} - 1}}{\Gamma(\alpha_k - \alpha_k^{n-1})} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_s^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_s^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds \\
&+ \frac{\Gamma(n - \mu_k) t^{n-1-\alpha_k^{n-1}}}{\Gamma(n - \alpha_k^{n-1}) (\Gamma(n - \mu_k) - 1)} \\
&\times \int_0^1 \frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_s^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_s^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds. \quad (7)
\end{aligned}$$

Theorem 2.1 Assume that hypotheses (H_1) and (H_2) are satisfied. Then, system (1) has a unique solution on $[0, 1]$.

Proof. We shall show that T is contractive operator on B .

Let $(u_1, \dots, u_n), (v_1, \dots, v_n) \in B$ and $t \in [0, 1]$. Then,

$$\begin{aligned}
& \|T_k(u_1, \dots, u_n) - T_k(v_1, \dots, v_n)\|_\infty \\
& \leq \max_{t \in [0, 1]} \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left| \begin{aligned} & f_k \begin{pmatrix} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{pmatrix} \\ & - f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} \end{aligned} \right| ds \\
& + \frac{\Gamma(n - \mu_k)}{|\Gamma(n - \mu_k) - 1| \Gamma(n)} \max_{t \in [0, 1]} t^{n-1} \\
& \times \int_0^1 \frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} \left| \begin{aligned} & f_k \begin{pmatrix} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{pmatrix} \\ & - f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} \end{aligned} \right| ds.
\end{aligned}$$

It follows the hypothesis (H_1) that,

$$\begin{aligned}
& \|T_k(u_1, \dots, u_n) - T_k(v_1, \dots, v_n)\|_\infty \\
& \leq \left(\max_{t \in [0,1]} \frac{t^{\alpha_k}}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(n - \mu_k)}{|\Gamma(n - \mu_k) - 1| \Gamma(n) \Gamma(\alpha_k - \mu_k + 1)} \right) \\
& \quad \times \left(\begin{aligned} & \lambda_1^k \|u_1 - v_1\|_\infty + \dots + \lambda_n^k \|u_n - v_n\|_\infty \\ & + \lambda_{n+1}^k \left\| D^{\alpha_1^1}(u_1 - v_1) \right\|_\infty + \dots \\ & + \lambda_{2n-1}^k \left\| D^{\alpha_1^{n-1}}(u_1 - v_1) \right\|_\infty + \dots \\ & + \lambda_{n+(n-1)^2+1}^k \left\| D^{\alpha_n^1}(u_n - v_n) \right\|_\infty + \dots \\ & + \lambda_{n^2}^k \left\| D^{\alpha_n^{n-1}}(u_n - v_n) \right\|_\infty \\ & + \lambda_{n^2+1}^k \omega_1 \|u_1 - v_1\|_\infty \max_{s \in [0,1]} \int_0^s d\tau + \dots \\ & + \lambda_{n^2+n}^k \omega_n \|u_n - v_n\|_\infty \max_{s \in [0,1]} \int_0^s d\tau \end{aligned} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|T_k(u_1, \dots, u_n) - T_k(v_1, \dots, v_n)\|_\infty \\
& \leq \left(\frac{1}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(n - \mu_k)}{|\Gamma(n - \mu_k) - 1| \Gamma(n) \Gamma(\alpha_k - \mu_k + 1)} \right) \\
& \quad \times \left(\sum_{j=1}^{n^2} \lambda_j^k + \sum_{j=1}^n \lambda_{n^2+j}^k \omega_j \right) \\
& \quad \times \max \left(\begin{aligned} & \|u_1 - v_1\|_\infty, \dots, \|u_n - v_n\|_\infty, \\ & \left\| D^{\alpha_1^1}(u_1 - v_1) \right\|_\infty, \dots, \left\| D^{\alpha_1^{n-1}}(u_1 - v_1) \right\|_\infty, \dots, \\ & \left\| D^{\alpha_n^1}(u_n - v_n) \right\|_\infty, \dots, \left\| D^{\alpha_n^{n-1}}(u_n - v_n) \right\|_\infty \end{aligned} \right).
\end{aligned}$$

Thus,

$$\|T_k(u_1, \dots, u_n) - T_k(v_1, \dots, v_n)\|_\infty \leq M_k \Delta_k \|u_1 - v_1, \dots, u_n - v_n\|_B. \quad (8)$$

By the same arguments, we get

$$\begin{aligned} & \left\| D^{\alpha_k^i} T_k(u_1, \dots, u_n) - D^{\alpha_k^i} T_k(v_1, \dots, v_n) \right\|_{\infty} \\ & \leq \left(\max_{t \in [0,1]} \frac{t^{\alpha_k - \alpha_k^i + 1}}{\Gamma(\alpha_k - \alpha_k^i + 1)} \right. \\ & \quad \left. + \frac{\Gamma(n - \mu_k)}{\Gamma(n - \alpha_k^i) |\Gamma(n - \mu_k) - 1| \Gamma(\alpha_k - \mu_k + 1)} \right) \\ & \quad \times \left(\begin{aligned} & \lambda_1^k \|u_1 - v_1\|_{\infty} + \dots + \lambda_n^k \|u_n - v_n\|_{\infty} \\ & + \lambda_{n+1}^k \left\| D^{\alpha_1^1}(u_1 - v_1) \right\|_{\infty} + \dots \\ & + \lambda_{2n-1}^k \left\| D^{\alpha_1^{n-1}}(u_1 - v_1) \right\|_{\infty} + \dots \\ & + \lambda_{n+(n-1)^2+1}^k \left\| D^{\alpha_n^1}(u_n - v_n) \right\|_{\infty} + \dots \\ & + \lambda_{n^2}^k \left\| D^{\alpha_n^{n-1}}(u_n - v_n) \right\|_{\infty} \\ & + \lambda_{n^2+1}^k \omega_1 \|u_1 - v_1\|_{\infty} \max_{s \in [0,1]} \int_0^s d\tau + \dots \\ & + \lambda_{n^2+n}^k \omega_n \|u_n - v_n\|_{\infty} \max_{s \in [0,1]} \int_0^s d\tau \end{aligned} \right). \end{aligned}$$

Then,

$$\begin{aligned} & \left\| D^{\alpha_k^i} T_k(u_1, \dots, u_n) - D^{\alpha_k^i} T_k(v_1, \dots, v_n) \right\|_{\infty} \\ & \leq \left(\frac{1}{\Gamma(\alpha_k - \alpha_k^i + 1)} \right. \\ & \quad \left. + \frac{\Gamma(n - \mu_k)}{\Gamma(n - \alpha_k^i) |\Gamma(n - \mu_k) - 1| \Gamma(\alpha_k - \mu_k + 1)} \right) \left(\sum_{j=1}^{n^2} \lambda_j^k + \sum_{j=1}^n \lambda_{n^2+j}^k \omega_j \right) \\ & \quad \times \max \left(\begin{aligned} & \|u_1 - v_1\|_{\infty}, \dots, \|u_n - v_n\|_{\infty}, \\ & \left\| D^{\alpha_1^1}(u_1 - v_1) \right\|_{\infty}, \dots, \left\| D^{\alpha_1^{n-1}}(u_1 - v_1) \right\|_{\infty}, \dots, \\ & \left\| D^{\alpha_n^1}(u_n - v_n) \right\|_{\infty}, \dots, \left\| D^{\alpha_n^{n-1}}(u_n - v_n) \right\|_{\infty} \end{aligned} \right). \end{aligned}$$

Thus, for all $k = 1, \dots, n$, and all $i = 1, \dots, n - 1$,

$$\begin{aligned} & \left\| D^{\alpha_k^i} T_k(u_1, \dots, u_n) - D^{\alpha_k^i} T_k(v_1, \dots, v_n) \right\|_{\infty} \\ & \leq M_k^i \Delta_k \|u_1 - v_1, \dots, u_n - v_n\|_B. \end{aligned} \tag{9}$$

By combining (8) with (9), we obtain:

$$\begin{aligned} & \|T(u_1, \dots, u_n) - T(v_1, \dots, v_n)\|_B \\ & \leq \max_{\substack{1 \leq k \leq n \\ 1 \leq i \leq n-1}} (M_k, M_k^i) \Delta_k \|u_1 - v_1, \dots, u_n - v_n\|_B. \end{aligned}$$

By the hypothesis (H_2) , we deduce that T is contractive. Using Banach fixed point theorem, we state that T has a fixed point which is the unique solution of (1). This completes the proof.

Example 2.2 Consider the following fractional system:

$$\left\{ \begin{array}{l}
 D^{\frac{9}{4}}u_1(t) = \\
 \quad \frac{\left| \begin{array}{l} u_1(t) + u_2(t) + u_3(t) + D^{\frac{1}{4}}u_1(t) + D^{\frac{7}{4}}u_1(t) \\ + D^{\frac{5}{6}}u_2(t) + D^{\frac{7}{6}}u_2(t) + D^{\frac{2}{3}}u_3(t) + D^{\frac{4}{3}}u_3(t) \end{array} \right|}{648\pi \left(1 + \left| \begin{array}{l} u_1(t) + u_2(t) + u_3(t) + D^{\frac{1}{4}}u_1(t) + D^{\frac{7}{4}}u_1(t) \\ + D^{\frac{5}{6}}u_2(t) + D^{\frac{7}{6}}u_2(t) + D^{\frac{2}{3}}u_3(t) + D^{\frac{4}{3}}u_3(t) \end{array} \right| \right)} \\
 \quad + \frac{t}{36} \left(\int_0^t \frac{\cos u_1(\tau)}{2\pi} d\tau + \int_0^t \frac{\sin u_2(\tau)}{2\pi + \tau} d\tau + \int_0^t \frac{\cos u_3(\tau)}{2\pi} d\tau \right), \\
 \\
 D^{\frac{13}{6}}u_2(t) = \\
 \quad \frac{\sin \left(\begin{array}{l} u_1(t) + u_2(t) + u_3(t) + D^{\frac{1}{4}}u_1(t) + D^{\frac{7}{4}}u_1(t) \\ + D^{\frac{5}{6}}u_2(t) + D^{\frac{7}{6}}u_2(t) + D^{\frac{2}{3}}u_3(t) + D^{\frac{4}{3}}u_3(t) \end{array} \right)}{324\pi} \\
 \quad + \frac{t^2+1}{72} \left(\int_0^t \frac{\cos u_1(\tau)}{2\pi} d\tau + \int_0^t \frac{\sin u_2(\tau)}{2\pi + \tau} d\tau + \int_0^t \frac{\cos u_3(\tau)}{2\pi} d\tau \right), \\
 \\
 D^{\frac{7}{3}}u_3(t) = \\
 \quad \frac{\left(\begin{array}{l} \cos \left(u_1(t) + D^{\frac{1}{4}}u_1(t) + D^{\frac{7}{4}}u_1(t) \right) \\ + \cos \left(u_2(t) + D^{\frac{5}{6}}u_2(t) + D^{\frac{7}{6}}u_2(t) \right) \\ + \cos \left(u_3(t) + D^{\frac{2}{3}}u_3(t) + D^{\frac{4}{3}}u_3(t) \right) \end{array} \right)}{162\pi} \\
 \quad + \frac{\pi}{36(\pi+t)} \left(\int_0^t \frac{\cos u_1(\tau)}{2\pi} d\tau + \int_0^t \frac{\sin u_2(\tau)}{2\pi + \tau} d\tau + \int_0^t \frac{\cos u_3(\tau)}{2\pi} d\tau \right), \\
 \\
 t \in [0, 1], \\
 u_1(0) = \sqrt{2}, \quad u_1'(0) = 1, \quad u_1''(1) = D^{\frac{5}{4}}u_1(1), \\
 u_2(0) = \sqrt{3}, \quad u_2'(0) = \sqrt{7}, \quad u_2''(1) = D^{\frac{4}{3}}u_2(1), \\
 u_3(0) = 7\sqrt{3}, \quad u_3'(0) = \sqrt{3}, \quad u_3''(1) = D^{\frac{5}{3}}u_3(1).
 \end{array} \right. \quad (10)$$

We have

$$\begin{aligned}
& u_k(t) \\
&= \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k \left(\begin{array}{l} s, u_1(s), u_2(s), u_3(s), D^{\alpha_1^1} u_1(s), D^{\alpha_1^2} u_1(s), \\ D^{\alpha_2^1} u_2(s), D^{\alpha_2^2} u_2(s), D^{\alpha_3^1} u_3(s), D^{\alpha_3^2} u_3(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \int_0^s h_2(\tau, u_2(\tau)) d\tau, \int_0^s h_3(\tau, u_3(\tau)) d\tau \end{array} \right) ds \\
&+ \sum_{j=0}^1 \frac{a_j^k}{j!} t^j + \frac{\Gamma(3-\mu_k) t^2}{(\Gamma(3-\mu_k)-1)\Gamma(3)} \\
&\times \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} f_k \left(\begin{array}{l} s, u_1(s), u_2(s), u_3(s), D^{\alpha_1^1} u_1(s), D^{\alpha_1^2} u_1(s), \\ D^{\alpha_2^1} u_2(s), D^{\alpha_2^2} u_2(s), D^{\alpha_3^1} u_3(s), D^{\alpha_3^2} u_3(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \int_0^s h_2(\tau, u_2(\tau)) d\tau, \int_0^s h_3(\tau, u_3(\tau)) d\tau \end{array} \right) ds,
\end{aligned}$$

where, $n = 3$, $k = 1, 2, 3$, $\alpha_1 = \frac{9}{4}$, $\alpha_2 = \frac{13}{6}$, $\alpha_3 = \frac{7}{3}$, $\alpha_1^1 = \frac{1}{4}$, $\alpha_1^2 = \frac{7}{4}$, $\alpha_2^1 = \frac{5}{6}$, $\alpha_2^2 = \frac{7}{6}$, $\alpha_3^1 = \frac{2}{3}$, $\alpha_3^2 = \frac{4}{3}$, $\mu_1 = \frac{5}{4}$, $\mu_2 = \frac{4}{3}$, $\mu_3 = \frac{5}{3}$, $a_0^1 = \sqrt{2}$, $a_1^1 = 1$, $a_0^2 = \sqrt{3}$, $a_1^2 = \sqrt{7}$, $a_0^3 = 7\sqrt{3}$, $a_1^3 = \sqrt{3}$, $J = [0, 1]$,

$$\begin{aligned}
h_1(\tau, u_1(\tau)) &= \frac{\cos u_1(\tau)}{2\pi}, \quad h_2(\tau, u_2(\tau)) = \frac{\sin u_2(\tau)}{2\pi + \tau}, \\
h_3(\tau, u_3(\tau)) &= \frac{\cos u_3(\tau)}{2\pi}.
\end{aligned}$$

For all $t \in [0, 1]$ and $(x_1, \dots, x_{12}), (y_1, \dots, y_{12}) \in \mathbb{R}^{12}$, we get

$$\begin{aligned}
& |f_1(t, x_1, x_2, \dots, x_{12}) - f_1(t, y_1, y_2, \dots, y_{12})| \\
&\leq \frac{1}{648\pi} \left(\begin{array}{l} |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4| \\ + |x_5 - y_5| + |x_6 - y_6| + |x_7 - y_7| \\ + |x_8 - y_8| + |x_9 - y_9| \end{array} \right) \\
&+ \frac{1}{36} \left(\frac{1}{2\pi} |x_{10} - y_{10}| + \frac{1}{2\pi} |x_{11} - y_{11}| + \frac{1}{2\pi} |x_{12} - y_{12}| \right),
\end{aligned}$$

$$\begin{aligned}
& |f_2(t, x_1, x_2, \dots, x_{12}) - f_2(t, y_1, y_2, \dots, y_{12})| \\
&\leq \frac{1}{324\pi} \left(\begin{array}{l} |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4| \\ + |x_5 - y_5| + |x_6 - y_6| + |x_7 - y_7| \\ + |x_8 - y_8| + |x_9 - y_9| \end{array} \right) \\
&+ \frac{1}{36} \left(\frac{1}{2\pi} |x_{10} - y_{10}| + \frac{1}{2\pi} |x_{11} - y_{11}| + \frac{1}{2\pi} |x_{12} - y_{12}| \right),
\end{aligned}$$

and

$$\begin{aligned}
& |f_3(t, x_1, x_2, \dots, x_{12}) - f_3(t, y_1, y_2, \dots, y_{12})| \\
& \leq \frac{1}{162\pi} \left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4| \right. \\
& \quad \left. + |x_5 - y_5| + |x_6 - y_6| + |x_7 - y_7| \right. \\
& \quad \left. + |x_8 - y_8| + |x_9 - y_9| \right) \\
& \quad + \frac{1}{36} \left(\frac{1}{2\pi} |x_{10} - y_{10}| + \frac{1}{2\pi} |x_{11} - y_{11}| + \frac{1}{2\pi} |x_{12} - y_{12}| \right).
\end{aligned}$$

So, we can take:

$$\begin{aligned}
(\lambda_j^1)_{j=1,2,\dots,9} &= \frac{1}{648\pi}, \quad (\lambda_j^2)_{j=1,2,\dots,9} = \frac{1}{324\pi}, \\
(\lambda_j^3)_{j=1,2,\dots,9} &= \frac{1}{162\pi}, \quad (\lambda_j^k)_{j=10,11,12}^{k=1,2,3} = \frac{1}{36}, \\
\omega_1 = \omega_2 = \omega_3 &= \frac{1}{2\pi}, \quad \Delta_1 = 0.0177, \Delta_2 = 0.0221, \quad \Delta_3 = 0.0309.
\end{aligned}$$

$$\begin{aligned}
M_1 &= 6.0699, \quad M_1^1 = 7.5601, \quad M_1^2 = 13.6562, \\
M_2 &= 5.3603, \quad M_2^1 = 9.9571, \quad M_2^2 = 11.4904, \\
M_3 &= 4.9815, \quad M_3^1 = 8.4276, \quad M_3^2 = 11.2387.
\end{aligned}$$

Indeed,

$$\begin{aligned}
\Delta_1 M_1 &= 0.1074, \quad \Delta_1 M_1^1 = 0.1338, \quad \Delta_1 M_1^2 = 0.2417, \\
\Delta_2 M_2 &= 0.1185, \quad \Delta_2 M_2^1 = 0.2201, \quad \Delta_2 M_2^2 = 0.2539, \\
\Delta_3 M_3 &= 0.1539, \quad \Delta_3 M_3^1 = 0.2604, \quad \Delta_3 M_3^2 = 0.3473.
\end{aligned}$$

Using Theorem 2.1, we deduce that the problem (10) has a unique solution on J .

Theorem 2.3 *Let the hypothesis (H_3) holds. Then, system (1) has at least one solution on J .*

Proof. *Let given the proof through two steps:*

Step 1: *We show that the operator T is completely continuous.*

Let $\lambda > 0$, we consider the set $\Omega_\lambda := \{(u_1, \dots, u_n) \in B : \|(u_1, \dots, u_n)\|_B \leq \lambda\}$, and we show that $T(\Omega_\lambda)$ is bounded.

For all $(u_1, \dots, u_n) \in \Omega_\lambda$, and all $t \in [0, 1]$, using hypothesis (H_3) , we have h_k are continuous on $[0, 1] \times [-\lambda, \lambda]$. Then, there exist $c_k > 0$:

$$|h_k(t, u_k(t))| \leq c_k, \quad k = 1, 2, \dots, n.$$

Also we get f_k are continuous on $[0, 1] \times [-\lambda, \lambda]^{n^2} \times [-c_1, c_1] \times \dots \times [-c_n, c_n]$, so, there exist $L_k > 0$:

$$\left| f_k \begin{pmatrix} t, u_1(t), \dots, u_n(t), D^{a_1^1} u_1(t), \dots, D^{a_1^{n-1}} u_1(t), \\ D^{a_2^1} u_2(t), \dots, D^{a_2^{n-1}} u_2(t), \dots, \\ D^{a_n^1} u_n(t), \dots, D^{a_n^{n-1}} u_n(t), \\ \int_0^t h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^t h_n(\tau, u_n(\tau)) d\tau \end{pmatrix} \right| \leq L_k. \quad (11)$$

Thanks to (11), we obtain

$$\begin{aligned} & \|T_k(u_1, \dots, u_n)\|_\infty \\ & \leq \sum_{j=0}^{n-2} \frac{|a_j^k|}{j!} + \left(\frac{1}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(n - \mu_k)}{|\Gamma(n - \mu_k) - 1| \Gamma(n) \Gamma(\alpha_k - \mu_k + 1)} \right) \\ & \quad \times \max_{s \in J} \left| f_k \begin{pmatrix} t, u_1(t), \dots, u_n(t), D^{a_1^1} u_1(t), \dots, D^{a_1^{n-1}} u_1(t), \\ D^{a_2^1} u_2(t), \dots, D^{a_2^{n-1}} u_2(t), \dots, \\ D^{a_n^1} u_n(t), \dots, D^{a_n^{n-1}} u_n(t), \\ \int_0^t h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^t h_n(\tau, u_n(\tau)) d\tau \end{pmatrix} \right| \\ & \leq \sum_{j=0}^{n-2} \frac{|a_j^k|}{j!} + L_k \left(\frac{1}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(n - \mu_k)}{|\Gamma(n - \mu_k) - 1| \Gamma(n) \Gamma(\alpha_k - \mu_k + 1)} \right). \end{aligned}$$

Therefore,

$$\|T_k(u_1, \dots, u_n)\|_\infty \leq \sum_{j=0}^{n-2} \frac{|a_j^k|}{j!} + L_k M_k. \quad (12)$$

On the other hand, we get

$$\begin{aligned}
& \left\| D^{\alpha_k^i} T_k(u_1, \dots, u_n) \right\|_{\infty} \\
& \leq \left(\frac{1}{\Gamma(\alpha_k - \alpha_k^i + 1)} + \frac{\Gamma(n - \mu_k)}{\Gamma(n - \alpha_k^i) |\Gamma(n - \mu_k) - 1| \Gamma(\alpha_k - \mu_k + 1)} \right) \\
& \quad \times \max_{s \in J} \left| f_k \begin{pmatrix} t, u_1(t), \dots, u_n(t), D^{a_1^1} u_1(t), \dots, D^{a_1^{n-1}} u_1(t), \\ D^{a_2^1} u_2(t), \dots, D^{a_2^{n-1}} u_2(t), \dots, \\ D^{a_n^1} u_n(t), \dots, D^{a_n^{n-1}} u_n(t), \\ \int_0^t h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^t h_n(\tau, u_n(\tau)) d\tau \end{pmatrix} \right| \\
& \quad + \sum_{j=i}^{n-2} \frac{|a_j^k|}{\Gamma(j+1 - \alpha_k^i)} \\
& \leq L_k \left(\frac{1}{\Gamma(\alpha_k - \alpha_k^i + 1)} + \frac{\Gamma(n - \mu_k)}{\Gamma(n - \alpha_k^i) |\Gamma(n - \mu_k) - 1| \Gamma(\alpha_k - \mu_k + 1)} \right) \\
& \quad + \sum_{j=i}^{n-2} \frac{|a_j^k|}{\Gamma(j+1 - \alpha_k^i)}.
\end{aligned}$$

Then,

$$\begin{aligned}
& \left\| D^{\alpha_k^i} T_k(u_1, \dots, u_n) \right\|_{\infty} \\
& \leq \sum_{j=i}^{n-2} \frac{|a_j^k|}{\Gamma(j+1 - \alpha_k^i)} + L_k M_k^i, \quad i = 1, 2, \dots, n-2.
\end{aligned} \tag{13}$$

Similarly, we obtain

$$\left\| D^{\alpha_k^{n-1}} T_k(u_1, \dots, u_n) \right\|_{\infty} \leq L_k M_k^{n-1}. \tag{14}$$

From (12), (13) and (14), we conclude that

$$\begin{aligned}
& \|T(u_1, \dots, u_n)\|_B \\
& \leq \max \left(\sum_{j=0}^{n-2} \frac{|a_j^k|}{j!} + L_k M_k, \sum_{j=i}^{n-2} \frac{|a_j^k|}{\Gamma(j+1 - \alpha_k^i)} + L_k M_k^i, L_k M_k^{n-1} \right) \\
& < \infty.
\end{aligned} \tag{15}$$

Hence, T maps bounded sets into bounded sets in B . The continuity of the functions f_k given in (H_3) implies that the operator T is continuous on B .

Let $0 \leq t_1 < t_2 \leq 1$, and $(u_1, \dots, u_n) \in \Omega_\lambda$. Therefore,

$$\begin{aligned} & |T_k(u_1, \dots, u_n)(t_2) - T_k(u_1, \dots, u_n)(t_1)| \\ & \leq \left(\frac{L_k}{\Gamma(\alpha_k+1)} (2(t_2 - t_1)^{\alpha_k} + (t_2^{\alpha_k} - t_1^{\alpha_k})) \right. \\ & \quad \left. + \sum_{j=0}^{n-2} \frac{|a_j^k| (t_2^j - t_1^j)}{j!} + \frac{L_k \Gamma(n - \mu_k) (t_2^{n-1} - t_1^{n-1})}{|\Gamma(n - \mu_k) - 1| \Gamma(n) \Gamma(\alpha_k - \mu_k + 1)} \right), \end{aligned} \quad (16)$$

$$\begin{aligned} & |D^{\alpha_k^i} T_k(u_1, \dots, u_n)(t_2) - D^{\alpha_k^i} T_k(u_1, \dots, u_n)(t_1)| \\ & \leq \left(\frac{L_k}{\Gamma(\alpha_k - \alpha_k^i + 1)} \left(2(t_2 - t_1)^{\alpha_k - \alpha_k^i} + (t_2^{\alpha_k - \alpha_k^i} - t_1^{\alpha_k - \alpha_k^i}) \right) \right. \\ & \quad \left. + \sum_{j=i}^{n-2} \frac{|a_j^k| (t_2^{j - \alpha_k^i} - t_1^{j - \alpha_k^i})}{\Gamma(j + 1 - \alpha_k^i)} + \frac{L_k \Gamma(n - \mu_k) (t_2^{n-1 - \alpha_k^i} - t_1^{n-1 - \alpha_k^i})}{\Gamma(n - \alpha_k^i) |\Gamma(n - \mu_k) - 1| \Gamma(\alpha_k - \mu_k + 1)} \right), \end{aligned} \quad (17)$$

where $i = 1, 2, \dots, n - 2$, and

$$\begin{aligned} & |D^{\alpha_k^{n-1}} T_k(u_1, \dots, u_n)(t_2) - D^{\alpha_k^{n-1}} T_k(u_1, \dots, u_n)(t_1)| \\ & \leq \left(\frac{L_k}{\Gamma(\alpha_k - \alpha_k^{n-1} + 1)} \left(2(t_2 - t_1)^{\alpha_k - \alpha_k^{n-1}} + (t_2^{\alpha_k - \alpha_k^{n-1}} - t_1^{\alpha_k - \alpha_k^{n-1}}) \right) \right. \\ & \quad \left. + \frac{L_k \Gamma(n - \mu_k) (t_2^{n-1 - \alpha_k^{n-1}} - t_1^{n-1 - \alpha_k^{n-1}})}{\Gamma(n - \alpha_k^{n-1}) |\Gamma(n - \mu_k) - 1| \Gamma(\alpha_k - \mu_k + 1)} \right). \end{aligned} \quad (18)$$

The right-hand sides of the inequalities (16), (17) and (18) are independent of $(u_1, \dots, u_n)$ and tend to zero as $t_2 - t_1 \rightarrow 0$. Consequently, T is an equi-continuous operator.

Thus, $T : B \rightarrow B$, is completely continuous.

Step 2: We shall show that

$$F := \{(u_1, \dots, u_n) \in B : (u_1, \dots, u_n) = \eta T(u_1, \dots, u_n), 0 < \eta < 1\},$$

is bounded.

For $(u_1, \dots, u_n) \in F$ and $t \in J$, we have

$$(u_1, \dots, u_n)(t) = \eta T(u_1, \dots, u_n)(t).$$

The inequality (15), yields

$$\begin{aligned} & \|(u_1, \dots, u_n)\|_B \\ & \leq \eta \max \left(\sum_{j=0}^{n-2} \frac{|a_j^k|}{j!} + L_k M_k, \sum_{j=i}^{n-2} \frac{|a_j^k|}{\Gamma(j + 1 - \alpha_k^i)} + L_k M_k^i, L_k M_k^{n-1} \right) \\ & < \infty. \end{aligned} \quad (19)$$

Hence, F is bounded.

Using Lemma 1.4, we state that system (1) has at least one solution on J . Theorem 2.3 is thus proved.

Example 2.4 Consider the fractional system

$$\left\{ \begin{array}{l}
 D^{\frac{7}{2}} u_1(t) = \\
 \frac{e^t \cos \left(u_1(t) + D^{\frac{1}{2}} u_1(t) + D^{\frac{3}{2}} u_1(t) + D^{\frac{5}{2}} u_1(t) \right)}{+ u_2(t) + D^{\frac{2}{3}} u_2(t) + D^{\frac{4}{3}} u_2(t) + D^{\frac{7}{3}} u_2(t)} \\
 \frac{2\pi + \sin \left(u_3(t) + D^{\frac{3}{4}} u_3(t) + D^{\frac{5}{4}} u_3(t) + D^{\frac{11}{4}} u_3(t) \right)}{+ u_4(t) + D^{\frac{1}{6}} u_4(t) + D^{\frac{11}{6}} u_4(t) + D^{\frac{13}{6}} u_4(t)} \\
 + \frac{2}{\pi} \left(\int_0^t \frac{\sin u_1(\tau)}{\tau^2 + \pi} d\tau + \int_0^t \frac{\cos u_2(\tau)}{e + \tau} d\tau + \int_0^t \frac{\sin u_3(\tau)}{(\tau + e^\tau)} d\tau \right), \\
 \\
 D^{\frac{11}{3}} u_2(t) = \\
 \frac{\cos \left(u_1(t) D^{\frac{1}{2}} u_1(t) D^{\frac{3}{2}} u_1(t) D^{\frac{5}{2}} u_1(t) \right) \sin \left(u_2(t) D^{\frac{2}{3}} u_2(t) D^{\frac{4}{3}} u_2(t) D^{\frac{7}{3}} u_2(t) \right)}{e - \cos \left(u_3(t) D^{\frac{3}{4}} u_3(t) D^{\frac{5}{4}} u_3(t) D^{\frac{11}{4}} u_3(t) + u_4(t) D^{\frac{1}{6}} u_4(t) D^{\frac{11}{6}} u_4(t) D^{\frac{13}{6}} u_4(t) \right)} \\
 + \frac{t}{\pi + e} \left(\int_0^t \frac{\sin u_1(\tau)}{\tau^2 + \pi} d\tau + \int_0^t \frac{\cos u_2(\tau)}{e + \tau} d\tau + \int_0^t \frac{\sin u_3(\tau)}{(\tau + e^\tau)} d\tau \right), \\
 \\
 D^{\frac{13}{4}} u_3(t) = \\
 \frac{\sin \left(u_3(t) D^{\frac{3}{4}} u_3(t) D^{\frac{5}{4}} u_3(t) D^{\frac{11}{4}} u_3(t) + u_4(t) D^{\frac{1}{6}} u_4(t) D^{\frac{11}{6}} u_4(t) D^{\frac{13}{6}} u_4(t) \right)}{2e^t - \cos \left(u_1(t) D^{\frac{1}{2}} u_1(t) D^{\frac{3}{2}} u_1(t) D^{\frac{5}{2}} u_1(t) + u_2(t) D^{\frac{2}{3}} u_2(t) D^{\frac{4}{3}} u_2(t) D^{\frac{7}{3}} u_2(t) \right)} \\
 + (t^2 + 1) \left(\int_0^t \frac{\sin u_1(\tau)}{\tau^2 + \pi} d\tau + \int_0^t \frac{\cos u_2(\tau)}{e + \tau} d\tau + \int_0^t \frac{\sin u_3(\tau)}{(\tau + e^\tau)} d\tau \right), \\
 \\
 D^{\frac{19}{6}} u_4(t) = \\
 \frac{\cos \left(u_1(t) D^{\frac{1}{2}} u_1(t) D^{\frac{3}{2}} u_1(t) D^{\frac{5}{2}} u_1(t) - u_4(t) D^{\frac{1}{6}} u_4(t) D^{\frac{11}{6}} u_4(t) D^{\frac{13}{6}} u_4(t) \right)}{3(t+1) + \cos \left(u_2(t) D^{\frac{2}{3}} u_2(t) D^{\frac{4}{3}} u_2(t) D^{\frac{7}{3}} u_2(t) - u_3(t) D^{\frac{3}{4}} u_3(t) D^{\frac{5}{4}} u_3(t) D^{\frac{11}{4}} u_3(t) \right)} \\
 + \frac{2te^t}{4\pi^2} \left(\int_0^t \frac{\sin u_1(\tau)}{\tau^2 + \pi} d\tau + \int_0^t \frac{\cos u_2(\tau)}{e + \tau} d\tau + \int_0^t \frac{\sin u_3(\tau)}{(\tau + e^\tau)} d\tau \right), \\
 \\
 t \in [0, 1], \\
 u_1(0) = \sqrt{3}, \quad u_1'(0) = -1, \quad u_1''(0) = -\sqrt{2}, \\
 u_1'''(1) = D^{\frac{5}{2}} u_1(1), \\
 u_2(0) = \sqrt{2}, \quad u_2'(0) = -1, \quad u_2''(0) = 1, \\
 u_2'''(1) = D^{\frac{8}{3}} u_2(1), \\
 u_3(0) = 3\sqrt{2}, \quad u_3'(0) = \frac{1}{2}, \quad u_3''(0) = -\sqrt{3}, \\
 u_3'''(1) = D^{\frac{9}{4}} u_3(1), \\
 u_4(0) = 2\sqrt{3}, \quad u_4'(0) = \sqrt{2} + 1, \quad u_4''(0) = 2\sqrt{5}, \\
 u_4'''(1) = D^{\frac{7}{3}} u_4(1).
 \end{array} \right. \quad (20)$$

We have:

$$\begin{aligned}
& u_k(t) \\
&= \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_4(s), D^{\alpha_1^1} u_1(s), D^{\alpha_2^1} u_1(s), D^{\alpha_3^1} u_1(s), \\ \dots, D^{\alpha_4^1} u_4(s), D^{\alpha_2^2} u_4(s), D^{\alpha_3^2} u_4(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^s h_4(\tau, u_4(\tau)) d\tau \end{array} \right) ds \\
&+ \sum_{j=0}^2 \frac{a_j^k}{j!} t^j + \frac{\Gamma(4-\mu_k) t^3}{(\Gamma(4-\mu_k)-1)\Gamma(4)} \\
&\times \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} f_k \left(\begin{array}{l} s, u_1(s), \dots, u_4(s), D^{\alpha_1^1} u_1(s), D^{\alpha_2^1} u_1(s), D^{\alpha_3^1} u_1(s), \\ \dots, D^{\alpha_4^1} u_4(s), D^{\alpha_2^2} u_4(s), D^{\alpha_3^2} u_4(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^s h_4(\tau, u_4(\tau)) d\tau \end{array} \right) ds,
\end{aligned}$$

where, $n = 4$, $k = 1, 2, 3, 4$, $\alpha_1 = \frac{7}{2}$, $\alpha_2 = \frac{11}{3}$, $\alpha_3 = \frac{13}{4}$, $\alpha_4 = \frac{19}{6}$, $\alpha_1^1 = \frac{1}{2}$, $\alpha_1^2 = \frac{3}{2}$, $\alpha_1^3 = \frac{5}{2}$, $\alpha_2^1 = \frac{2}{3}$, $\alpha_2^2 = \frac{4}{3}$, $\alpha_2^3 = \frac{7}{3}$, $\alpha_3^1 = \frac{3}{4}$, $\alpha_3^2 = \frac{5}{4}$, $\alpha_3^3 = \frac{11}{4}$, $\alpha_4^1 = \frac{1}{6}$, $\alpha_4^2 = \frac{11}{6}$, $\alpha_4^3 = \frac{13}{6}$, $\mu_1 = \frac{5}{2}$, $\mu_2 = \frac{8}{3}$, $\mu_3 = \frac{9}{4}$, $\mu_4 = \frac{7}{3}$, $a_0^1 = \sqrt{3}$, $a_1^1 = -1$, $a_2^1 = -\sqrt{2}$, $a_0^2 = \sqrt{2}$, $a_1^2 = -1$, $a_2^2 = 1$, $a_0^3 = 3\sqrt{2}$, $a_1^3 = \frac{1}{2}$, $a_2^3 = -\sqrt{3}$, $a_0^4 = 2\sqrt{3}$, $a_1^4 = \sqrt{2} + 1$, $a_2^4 = 2\sqrt{5}$,

Since the assumption of Theorem 2.3 is satisfied, system (20) has at least one solution on J .

3 Generalized Ulam-Hyers Stability

In this section, we will define and study the Ulam-Hyers stability and the generalized Ulam-Hyers stability of system (1).

Definition 3.1 System (1) is Ulam-Hyers stable, if there exists a constant $\delta_{f_k} > 0$, such that for all $(\epsilon_1, \dots, \epsilon_n) > 0$, and for all solution $(v_1, \dots, v_n) \in B$ of

$$\left\{ \begin{array}{l} \left| D^{\alpha_k} v_k(t) - f_k \left(\begin{array}{l} t, v_1(t), \dots, v_n(t), D^{\alpha_1^1} v_1(t), \dots, D^{\alpha_{n-1}^1} v_1(t), \\ D^{\alpha_2^1} v_2(t), \dots, D^{\alpha_{n-1}^2} v_2(t), \dots, D^{\alpha_n^1} v_n(t), \dots, D^{\alpha_{n-1}^n} v_n(t), \\ \int_0^t h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^t h_n(\tau, v_n(\tau)) d\tau \end{array} \right) \right| \leq \epsilon_k, \\ t \in [0, 1], \quad n-1 < \alpha_k < n, \quad k = 1, 2, \dots, n, \\ i-1 < \alpha_k^i < i, \quad i = 1, 2, \dots, n-1, \\ v_k^{(j)}(0) = a_j^k, \quad v_k^{(n-1)}(1) = D^{\mu_k} v_k(1), \quad n-2 < \mu_k < n-1, \end{array} \right. \quad (21)$$

there exists $(u_1, \dots, u_n) \in B$ of (1), such that

$$\|(v_1 - u_1, \dots, v_n - u_n)\|_B \leq \delta_{f_k} \epsilon, \quad \epsilon > 0. \quad (22)$$

Definition 3.2 The fractional coupled system (1) is generalized Ulam-Hyers stable if there exists $\Upsilon_{f_k} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\Upsilon_{f_k}(0) = 0$, such that for all $\epsilon > 0$, and for each solution $(v_1, \dots, v_n) \in B$ of (21), there exists $(u_1, \dots, u_n) \in B$ of (1), with

$$\|(v_1 - u_1, \dots, v_n - u_n)\|_B \leq \Upsilon_{f_k}(\epsilon), \quad \epsilon > 0. \quad (23)$$

Remark 3.3 $(v_1, \dots, v_n) \in B$ is a solution of of Ineq. (21), if and only if there exist $h_k \in C([0, 1], \mathbb{R})$, $k = 1, \dots, n$, such that:

$$|h_k(t)| < \epsilon_k, \quad t \in [0, 1],$$

and

$$D^{\alpha_k} v_k(t) = f_k \left(\begin{array}{c} t, v_1(t), \dots, v_n(t), D^{a_1^1} v_1(t), \dots, D^{a_1^{n-1}} v_1(t), \\ D^{a_2^1} v_2(t), \dots, D^{a_2^{n-1}} v_2(t), \dots, D^{a_n^1} v_n(t), \dots, D^{a_n^{n-1}} v_n(t), \\ \int_t^t h_1(\tau, v_1(\tau)) d\tau, \dots, \int_t^t h_n(\tau, v_n(\tau)) d\tau \end{array} \right) + h_k(t).$$

Theorem 3.4 Let $n - 1 < \alpha_k < n$, $n \in \mathbb{N} - \{0, 1\}$, and the hypotheses $(H_i)_{i=1,2}$ hold. Then, system (1) is generalized Ulam-Hyers stable in B .

Proof. Let $(v_1, \dots, v_n) \in B$ be solution of Ineq. (21). Then, $(v_1, \dots, v_n) \in B$ is a solution of the following integral inequality:

$$\begin{aligned} & \left| v_k(t) - \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k \left(\begin{array}{c} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_s^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_s^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) ds \right. \\ & \quad \left. - \sum_{j=0}^{n-2} \frac{a_j^k}{j!} t^j - \frac{\Gamma(n-\mu_k) t^{n-1}}{(\Gamma(n-\mu_k)-1)\Gamma(n)} \right. \\ & \quad \left. \times \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} f_k \left(\begin{array}{c} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_s^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_s^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) ds \right| \\ & \leq J^{\alpha_k} \epsilon_k \\ & \leq \frac{t^{\alpha_k}}{\Gamma(\alpha_k + 1)} \epsilon_k, \quad k = 1, 2, \dots, n. \end{aligned} \quad (24)$$

Using (H_1) and (H_2) , there exists a solution $(u_1, \dots, u_n) \in B$ of Eq. (1). Then, we can write

$$\begin{aligned}
 & |v_k(t) - u_k(t)| \\
 & \left| v_k(t) - \sum_{j=0}^{n-2} \frac{a_k^j}{j!} t^j \right. \\
 & \quad - \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k \left(\begin{array}{c} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) ds \\
 & \quad - \frac{\Gamma(n-\mu_k)t^{n-1}}{(\Gamma(n-\mu_k)-1)\Gamma(n)} \\
 & \quad \times \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} f_k \left(\begin{array}{c} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) ds \\
 & \quad + \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left(\begin{array}{c} f_k \left(\begin{array}{c} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) \\ - f_k \left(\begin{array}{c} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) \end{array} \right) ds \\
 & \quad + \frac{\Gamma(n-\mu_k)t^{n-1}}{(\Gamma(n-\mu_k)-1)\Gamma(n)} \\
 & \quad \times \int_0^1 \frac{(1-s)^{\alpha_k-\mu_k-1}}{\Gamma(\alpha_k-\mu_k)} \left(\begin{array}{c} f_k \left(\begin{array}{c} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) \\ - f_k \left(\begin{array}{c} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) \end{array} \right) ds \Bigg|.
 \end{aligned} \tag{25}$$

By Ineq. (24), we get

$$\begin{aligned}
& \max_{t \in [0,1]} |v_k(t) - u_k(t)| \\
& \leq \frac{\epsilon_k}{\Gamma(\alpha_k + 1)} \\
& + \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left| f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} \right. \\
& \quad \left. - f_k \begin{pmatrix} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{pmatrix} \right| ds \\
& + \frac{\Gamma(n - \mu_k) t^{n-1}}{|\Gamma(n - \mu_k) - 1| \Gamma(n)} \\
& \times \int_0^1 \frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} \left| f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} \right. \\
& \quad \left. - f_k \begin{pmatrix} s, u_1(s), \dots, u_n(s), \\ D^{\alpha_1^1} u_1(s), \dots, D^{\alpha_1^{n-1}} u_1(s), \\ \dots, D^{\alpha_n^1} u_n(s), \dots, D^{\alpha_n^{n-1}} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{pmatrix} \right| ds.
\end{aligned} \tag{26}$$

Which implies that

$$\|v_k - u_k\|_\infty \leq \frac{\epsilon_k}{\Gamma(\alpha_k + 1)} + M_k \Delta_k \|(v_1 - u_1, \dots, v_n - u_n)\|_B. \tag{27}$$

On the other hand, Ineq. (24), yields

$$\begin{aligned}
& \left| D^{\alpha_k^i} v_k - \int_0^t \frac{(t-s)^{\alpha_k - \alpha_k^i - 1}}{\Gamma(\alpha_k - \alpha_k^i)} f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} ds \right. \\
& \quad \left. - \sum_{j=i}^{n-2} \frac{a_j^k}{\Gamma(j+1-\alpha_k^i)} t^{j-\alpha_k^i} - \frac{\Gamma(n-\mu_k) t^{n-1-\alpha_k^i}}{\Gamma(n-\alpha_k^i)(\Gamma(n-\mu_k)-1)} \right. \\
& \quad \left. \times \int_0^1 \frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} ds, \right| \\
& \leq J^{\alpha_k - \alpha_k^i} \epsilon_k \\
& \leq \frac{t^{\alpha_k - \alpha_k^i}}{\Gamma(\alpha_k - \alpha_k^i + 1)} \epsilon_k, \quad k = 1, 2, \dots, n, \quad i = 1, 2, \dots, n-2, \tag{28}
\end{aligned}$$

and

$$\begin{aligned}
& \left| D^{\alpha_k^{n-1}} v_k - \int_0^t \frac{(t-s)^{\alpha_k - \alpha_k^{n-1} - 1}}{\Gamma(\alpha_k - \alpha_k^{n-1})} f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \\ \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} ds \right. \\
& \quad \left. - \frac{\Gamma(n-\mu_k) t^{n-1-\alpha_k^{n-1}}}{\Gamma(n-\alpha_k^{n-1})(\Gamma(n-\mu_k)-1)} \right. \\
& \quad \left. \times \int_0^1 \frac{(1-s)^{\alpha_k - \mu_k - 1}}{\Gamma(\alpha_k - \mu_k)} f_k \begin{pmatrix} s, v_1(s), \dots, v_n(s), \\ D^{\alpha_1^1} v_1(s), \dots, D^{\alpha_1^{n-1}} v_1(s), \\ \dots, D^{\alpha_n^1} v_n(s), \dots, D^{\alpha_n^{n-1}} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau)) d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau)) d\tau \end{pmatrix} ds \right| \\
& \leq J^{\alpha_k - \alpha_k^{n-1}} \epsilon_k \\
& \leq \frac{t^{\alpha_k - \alpha_k^{n-1}}}{\Gamma(\alpha_k - \alpha_k^{n-1} + 1)} \epsilon_k, \quad k = 1, 2, \dots, n, \quad i = n-1. \tag{29}
\end{aligned}$$

Similarly as before, we can show that

$$\begin{aligned} \left\| D^{\alpha_k^i} (v_k - u_k) \right\|_{\infty} &\leq \frac{\epsilon_k}{\Gamma(\alpha_k - \alpha_k^i + 1)} + M_k^i \Delta_k \|(v_1 - u_1, \dots, v_n - u_n)\|_B, \\ k &= 1, 2, \dots, n, \quad i = 1, 2, \dots, n-1. \end{aligned} \quad (30)$$

It follows (27) and (30), that

$$\begin{aligned} &\|(v_1 - u_1, \dots, v_n - u_n)\|_B \\ &\leq \max_{\substack{1 \leq k \leq n \\ 1 \leq i \leq n-1}} \left(\frac{\epsilon_k}{\Gamma(\alpha_k + 1)}, \frac{\epsilon_k}{\Gamma(\alpha_k - \alpha_k^i + 1)} \right) \\ &\quad + \max_{\substack{1 \leq k \leq n \\ 1 \leq i \leq n-1}} (M_k \Delta_k, M_k^i \Delta_k) \|(v_1 - u_1, \dots, v_n - u_n)\|_B \\ &\leq \epsilon \Lambda + \Theta \|(v_1 - u_1, \dots, v_n - u_n)\|_B, \end{aligned} \quad (31)$$

where

$$\epsilon = \max_{1 \leq k \leq n} \epsilon_k, \quad \Lambda = \max_{\substack{1 \leq k \leq n \\ 1 \leq i \leq n-1}} \left(\frac{1}{\Gamma(\alpha_k + 1)}, \frac{1}{\Gamma(\alpha_k - \alpha_k^i + 1)} \right). \quad (32)$$

Hence,

$$\|(v_1 - u_1, \dots, v_n - u_n)\|_B \leq \frac{\epsilon \Lambda}{(1 - \Theta)} := \delta_{f_k} \epsilon, \quad \delta_{f_k} = \frac{\Lambda}{(1 - \Theta)}. \quad (33)$$

From (H_2) , we have $\delta_{f_k} > 0$. So, system (1) is Ulam-Hyers stable. Taking $\Upsilon_{f_k}(\epsilon) = \delta_{f_k} \epsilon$, we get system (1) is generalized Ulam-Hyers stable. This ends the proof.

4 Open Problems

1. What are the sufficient conditions that guarantee Ulam-Hyers Rassias stability for the problem considered in this paper?
2. The problem considered in this paper can be generalized to a higher dimension?

3. If $f_k : (0, 1] \times \mathbb{R}^{n^2+n} \rightarrow \mathbb{R}$ are continuous functions, singular at $t = 0$, and $\lim_{t \rightarrow 0^+} f_k(t) = \infty$, the open problem is to find sufficient conditions for the existence and uniqueness of solutions as well as generalized stability conditions in this case.

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