

A New Statistical Model for Extreme Values: Properties and Applications

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Abstract

In this work, a new three parameter extreme value model is defined and studied. Various structural properties of the proposed model are investigated. The new density is expressed as a linear mixture of the Fréchet density. Method of the maximum likelihood is implemented to estimate the model parameters. Simulations are conducted for different sample sizes to examine the performance of the maximum likelihood estimators for the model parameters. The new model is applied to

two real data sets to empirically prove its flexibility in comparison to some known models.

Keywords: *Extreme value models, Maximum likelihood, moments, Order statistics, Fréchet model.*

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1 Introduction

Assume that $Y_1, Y_2, \dots, Y_m, \dots$ is a sequence of *iid* (independent and identically distributed) random variables with common cdf (cumulative distribution function) $F(y)$. One of the most interesting statistics is the sample maximum $M_m = \max \{Y_1, Y_2, \dots, Y_m\}$. One is keen on the behaviour of M_m as the sample size m increases to infinity.

$$\begin{aligned} p_r \{M_m \leq y\} &= p_r \{Y_1 \leq y, Y_2 \leq y, \dots, Y_m \leq y, \} \\ &= p_r \{Y_1 \leq y\} \dots p_r \{Y_m \leq y\} \\ &= F(y)^m. \end{aligned}$$

Assume there are sequences of constants $\{\tau_m > 0\}$ and $\{\ell_m\}$ such that

$$p_r \left\{ \frac{M_m - \ell_m}{\tau_m} \leq y \right\} \rightarrow G(y) \text{ as } m \rightarrow \infty,$$

then if $G(y)$ is a non-degenerate distribution function, then it will belong to one of the three following fundamental types of classic extreme value family: which is Type **I** (Gumbel distribution); Type **II** (Fréchet distribution); Type **III** (Weibull distribution). The extreme value theory concentrates on the behavior of the block maxima or minima. The extreme value theory was presented first by Fréchet [10] and Fisher and Tippett [11], then took after by Von Mises [22, 23]. The so called Fréchet ('Fr' for short) distribution is one of the most important distributions in extreme value theory and it has applications ranging from accelerated life testing through earthquakes, rainfall, floods, horse racing, queues in supermarkets, wind speeds and sea waves.

For more details about the Fr distribution and its applications, see Kotz and Nadarajah [15]. Moreover, applications of this distribution in various fields are given in Harlow [13]. Of late, some new extensions of the Fréchet distribution were considered by Nadarajah and Kotz [19], Nadarajah and Gupta [18], Nadarajah and Kotz [20], Barreto-Souza et al. [5], Mahmoud and Mandouh [16], Krishna et.al. [14], da Silva et al. [8], Elbatal et al. [9], Afify et al. [1], Yousof et al. [24], Mead et al. [17], Afify et al. [2], Afify et al. [3], Afify et al. [4], Yousof et al. [25], Cordeiro et al. [6] and Yousof et al. [26].

The probability density function (pdf) and cumulative distribution function (cdf) of the Fréchet (Fr) distribution are given by (for $x \geq 0$)

$$g(x; \zeta, \theta) = \theta \zeta^\theta x^{-(\theta+1)} \exp \left[-(\zeta/x)^\theta \right] \text{ and } G(x, \zeta, \theta) = \exp \left[-(\zeta/x)^\theta \right], \quad (1)$$

respectively, where $\zeta > 0$ is a scale parameter and $\theta > 0$ is a shape parameter.

The aim of this paper is to propose the first generalization of the Fr model using the Zografos–Balakrishnan-G family of distributions which was introduced by Nadarajah et al. [21]. The pdf and cdf of the Zografos–Balakrishnan-G family of distributions are respectively given by

$$f(x; \lambda, \xi) = \Gamma^{-1}(\lambda) g(x; \xi) \left\{ -\log \overline{G}(x; \xi) \right\}^{\lambda-1}, \quad (2)$$

and

$$F(x; \lambda, \xi) = \Gamma^{-1}(\lambda) \int_0^{-\log \overline{G}(x; \xi)} t^{\lambda-1} \exp(-t) dt = \Gamma^{-1}(\lambda) \gamma(\lambda, -\log \overline{G}(x; \xi)). \quad (3)$$

We use equations (1) and (2) to obtain the three-parameter ZBFR pdf (for $x \geq 0$) as

$$f(x) = \theta \zeta^\theta \Gamma^{-1}(\lambda) x^{-(\theta+1)} \exp \left[-(\zeta/x)^\theta \right] \left(-\log \left\{ 1 - \exp \left[-(\zeta/x)^\theta \right] \right\} \right)^{\lambda-1}, \quad (4)$$

and the corresponding cdf and hazard rate function are given respectively by

$$F(x) = \Gamma^{-1}(\lambda) \gamma \left(\lambda, -\log \left\{ 1 - \exp \left[-(\zeta/x)^\theta \right] \right\} \right), \quad (5)$$

and

$$h(x) = \theta \zeta^\theta x^{-(\theta+1)} \exp \left[-(\zeta/x)^\theta \right] \gamma^{-1} \left(\lambda, -\log \left\{ 1 - \exp \left[-(\zeta/x)^\theta \right] \right\} \right), \quad (6)$$

where $\gamma^{-1}(\delta, z) = 1/\int_z^\infty t^{\delta-1} \exp(-t) dt$ is the complementary incomplete gamma function. Henceforth, we denote a random variable X having pdf (4) by $X \sim \text{ZBFR}(\lambda, \zeta, \theta)$. The importance of (4) is that it contains as sub-models several generated distributions. Following Nadarajah et al. [21] and expanding the quantity $\left(-\log \left\{ 1 - \exp \left[-(\zeta/x)^\theta \right] \right\} \right)^{\lambda-1}$ in (4) we get

$$f(x) = \sum_{k=0}^{\infty} \Psi_k \pi_{\zeta+k}(x), \quad \forall \zeta > 0 \text{ and } \zeta \in \mathbb{R}, \quad (7)$$

where

$$\Psi_k = \sum_{j=0}^k \frac{(-1)^{k+j+r} P_{j,k}}{(\lambda+k) \Gamma(\lambda-1)} \binom{k-\lambda+1}{k} \binom{k}{j},$$

and $\pi_{\zeta+k}(x)$ is the Fr density with scale parameter $a(\zeta+k)^{\frac{1}{b}}$ and shape parameter b , the constants $P_{j,k}$ can be calculated recursively by

$$P_{j,k} = k^{-1} \sum_{m=1}^k [k-m(j+1)] s_m \Upsilon_{j,k-m}$$

for $k = 1, 2, \dots$ with $\Upsilon_{j,0} = 1$ and $s_k = \frac{(-1)^{k+1}}{(k+1)}$, and

$$\pi_{\lambda+k}(x) = (\zeta+k) \theta \zeta^\theta x^{-(\theta+1)} \exp \left[-(\zeta/x)^\theta \right] \left\{ \exp \left[-(\zeta/x)^\theta \right] \right\}^{\lambda+k-1}$$

is the Fr density with with scale parameter $\zeta(\lambda+k)^{\frac{1}{\theta}}$ and shape parameter θ . Similarly, the cdf in (5) can be expressed as

$$F(x) = \sum_{k=0}^{\infty} \Psi_k \Pi_{\lambda+k}(x), \quad (8)$$

where $\Pi_{\lambda+k}(x) = \exp \left[-(\lambda+k) (\zeta/x)^\theta \right]$ is the Fr cdf with scale parameter $\zeta(\lambda+k)^{\frac{1}{\theta}}$ and shape parameter θ .

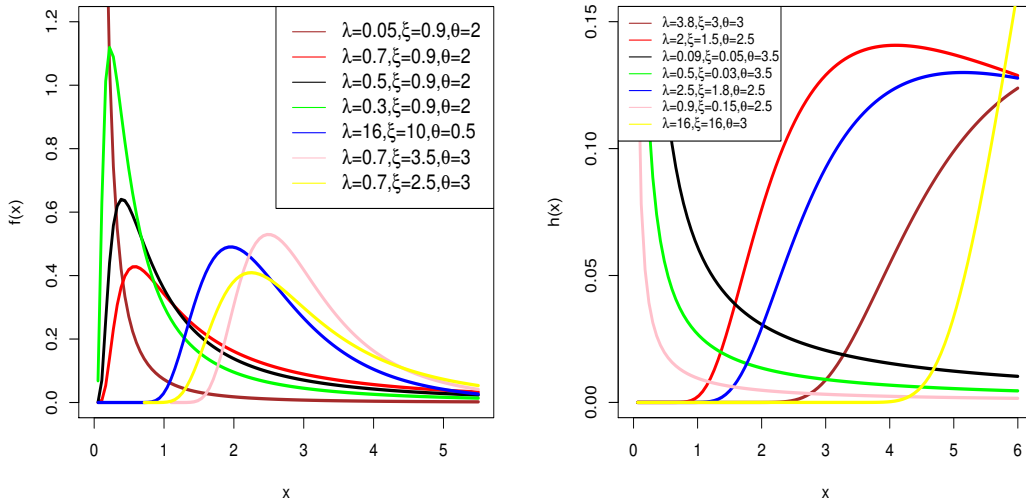


Figure 1: Plots of the ZBFR pdf and hrf for some parameter values.

The rest of the paper is outlined as follows. In Section 2, we derive some mathematical properties including moments, generating function, residual life and reversed residual life functions, stress-strength model and order statistics and their moments are introduced at the end of the section. Maximum likelihood estimation of the model parameters is addressed in Section 3. In section 4, simulation results to assess the performance of maximum likelihood estimators of the proposed distribution are discussed. We provide two applications to real data sets to illustrate the importance of the new model in Section 5. Finally, we offer some concluding remarks in Section 6.

2 Mathematical properties

In this section we provide some mathematical properties of the ZBFr distribution.

2.1 Moments and cumulants

The r th ordinary moment of X is given by

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} \Psi_k \int_{-\infty}^{\infty} x^r \pi_{\lambda+k}(x) dx,$$

then we obtain

$$\mu'_r = \sum_{k=0}^{\infty} \frac{\Psi_k}{\zeta^{-r}} (\lambda + k)^{r/\theta} \Gamma(1 - r/\theta), \forall r < \theta. \quad (9)$$

2.2 Moment generating function

Here, we provide two formulae for the moment generating function (mgf) $M_X(t) = E(e^{tX})$ of X . The first one can be derived from equation (9) as

$$M_X(t) = \sum_{k,r=0}^{\infty} \frac{\Psi_k}{\zeta^{-r}} \frac{t^r}{r!} (\lambda + k)^{r/\theta} \Gamma(1 - r/\theta), \forall r < \theta.$$

As for the second formula for $M_X(t)$, setting $y = x^{-1}$ in (1), we first derive the mgf of Fr distribution

$$M(t; \zeta, \theta) = \theta \zeta^\theta \int_0^\infty \exp(t/y) y^{(\theta-1)} \exp\left\{- (\zeta y)^\theta\right\} dy.$$

By expanding the first exponential and calculating the integral, we have

$$M(t; \zeta, \theta) = \theta \zeta^\theta \int_0^\infty \sum_{m=0}^\infty \frac{t^m}{m!} \exp(t/y) y^{\theta-m-1} \exp\left\{- (\zeta y)^\theta\right\} dy$$

$$= \sum_{m=0}^\infty \frac{\zeta^m t^m}{m!} \Gamma(1 - m/\theta),$$

where the gamma function is well-defined for any non-integer θ .

Consider the Wright generalized hypergeometric function defined by

$${}_p\Phi_q \left[\begin{matrix} (\zeta_1, A_1), \dots, (\zeta_p, A_p) \\ (\theta_1, B_1), \dots, (\theta_q, B_q) \end{matrix} ; x \right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p \Gamma(\zeta_j + A_j n)}{\prod_{j=1}^q \Gamma(\theta_j + B_j n)} \frac{x^n}{n!}.$$

Then, we have

$$M(t; \zeta, \theta) = {}_1\Phi_0 \left[\begin{matrix} (1, -\frac{1}{\theta}) \\ - \end{matrix} ; \zeta t \right],$$

Finally, we obtain the mgf of $X \sim ZBFr$, say $M_X(t)$, using (7) as

$$M_X(t) = \sum_{k=0}^\infty \Psi_k {}_1\Phi_0 \left[\begin{matrix} (1, -\frac{1}{\theta}) \\ - \end{matrix} ; \frac{\zeta}{(\lambda + k)^{-\frac{1}{\theta}}} t \right].$$

2.3 Incomplete moments and mean deviations

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The m th incomplete moment, say $I_m(t)$, of X can be expressed from (7) as

$$I_m(t) = \sum_{k=0}^\infty \Psi_k \int_{-\infty}^t x^m \pi_{\lambda+k}(x) dx$$

$$= \sum_{k=0}^\infty \frac{\Psi_k}{\zeta^{-m}} (\lambda + k)^{m/\theta} \gamma\left(1 - m/\theta, (\lambda + k) (\zeta/t)^\theta\right), \forall m < \theta.$$

The mean deviations about the mean [$\delta_1 = E(|X - \mu'_1|)$] and about the median [$\delta_2 = E(|X - M|)$] of X are given by $\delta_1 = 2\mu'_1 F(\mu'_1) - 2I_1(\mu'_1)$ and $\delta_2 = \mu'_1 - 2I_1(M)$, respectively, where $\mu'_1 = E(X)$, $M = \text{Median}(X) = Q(0.5)$ is the median, $F(\mu'_1)$ is easily calculated from (5) and $I_1(t)$ is the first incomplete moment given by the last equation with $m = 1$. A general equation for $I_1(t)$ can be derived from the last equation as

$$I_1(t) = \sum_{k=0}^\infty \frac{\Psi_k}{\zeta^{-1}} (\lambda + k)^{\theta-1} \gamma\left(1 - \theta^{-1}, (\lambda + k) (\zeta/t)^\theta\right).$$

2.4 Moment of residual and reversed residual life

The n th moment of the residual life, say $m_n(t) = E[(X - t)^n | X > t]$, $n = 1, 2, \dots$, uniquely determines $F(x)$. The n th moment of the residual life of X is given by $m_n(t) = \frac{1}{1-F(t)} \int_t^\infty (x - t)^n dF(x)$. Therefore,

$$m_n(t) = [1 - F(t)]^{-1} \sum_{k=0}^{\infty} \frac{\Psi_k^*}{\zeta^{-n}} (\lambda + k)^{n/\theta} \Gamma\left(1 - n/\theta, (\lambda + k)(\zeta/t)^\theta\right), \forall n < \theta,$$

where $\Psi_k^* = \Psi_k \sum_{r=0}^n (1 - t)^n$. Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X - t) | X > t]$, which represents the expected additional life length for a unit which is alive at age t . The MRL of X can be obtained by setting $n = 1$ in the last equation. The n th moment of the reversed residual life, say $M_n(t) = E[(t - X)^n | X \leq t]$ for $t > 0$ and $n = 1, 2, \dots$ uniquely determines $F(x)$. We obtain $M_n(t) = \frac{1}{F(t)} \int_0^t (t - x)^n dF(x)$. Then, the n th moment of the reversed residual life of X becomes

$$M_n(t) = F^{-1}(t) \sum_{k=0}^{\infty} \frac{\Psi_k^{**}}{\zeta^{-n}} (\lambda + k)^{n/\theta} \gamma\left(1 - n/\theta, (\lambda + k)(\zeta/t)^\theta\right), \forall n < \theta,$$

where $\Psi_k^{**} = \Psi_k \sum_{r=0}^n (-1)^r \binom{n}{r} t^{n-r}$. The mean inactivity time (MIT) or mean waiting time also called the mean reversed residual life function is given by $M_1(t) = E[(t - X) | X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$. The MIT of the ZBFR distribution can be obtained easily by setting $n = 1$ in the above equation.

2.5 Order statistics

Suppose $X_1, \dots, X_n$ is a random sample from any ZBFR distribution. Let $X_{i:n}$ denote the i^{th} order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} (-1)^j [B(i, n - i + 1)]^{-1} f(x) F(x)^{j+i-1} \binom{n-i}{j}.$$

Following similar algebraic developments of Nadarajah et al. [21], we can write the density function of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} \Psi_{r,k} \pi_{r+k+1}(x), \quad (11)$$

where

$$\Psi_{r,k} = \frac{n! (r+1) (i-1)! \Psi_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j \psi_{j+i-1,k}}{(n-i-j)! j!},$$

Ψ_k is given in Section 1 and the quantities $\psi_{j+i-1,k}$ can be determined with $\psi_{j+i-1,0} = \Psi_0^{j+i-1}$ and recursively

$$\psi_{j+i-1,k} = (k c_0)^{-1} \sum_{m=1}^k [m(j+i) - k] \Psi_m \psi_{j+i-1,k-m}, \quad \forall k \geq 1.$$

Equation (11) is the main result of this section. The q^{th} moment of i^{th} order statistic is given by

$$E(X_{i:n}^q) = \sum_{r,k=0}^{\infty} \frac{\Psi_{r,k}}{\zeta^{-q}} (\lambda + k + 1)^{q/\theta} \Gamma(1 - q/\theta), \quad \forall q < \theta.$$

3 Estimation

Several approaches for parameter estimation has been proposed in the literature but maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood. Let $x_1, \dots, x_n$ be a random sample from ZBFr distributions with parameters λ, θ, ζ , and θ . Let $\eta = (\lambda, \zeta, \theta)^\top$ be the 3×1 parameter vector. For determining the MLE of η , we have the log-likelihood function

$$\begin{aligned} \ell &= \ell(\eta) = n \log \theta + n \theta \log \zeta - n \log [\Gamma(\lambda)] - (\theta + 1) \sum_{i=1}^n \log x_i \\ &\quad - \sum_{i=1}^n (\zeta/x_i)^\theta + (\lambda - 1) \sum_{i=1}^n \log [-\log(1 - s_i)] \end{aligned}$$

where $s_i = \exp \left[-(\zeta/x_i)^\theta \right]$. The components of the score vector, $\mathbf{U}(\eta) = \left(\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \zeta}, \frac{\partial \ell}{\partial \theta} \right)^\top$, are

$$\begin{aligned} U_\lambda &= -n\psi(\lambda) + \sum_{i=1}^n \log [-\log(1 - s_i)], \\ U_\zeta &= \frac{n\theta}{\zeta} - \theta \sum_{i=1}^n \zeta^{\theta-1}/x_i^\theta - (\lambda - 1) \sum_{i=1}^n \frac{m_i(1-s_i)^{-1}}{\log(1-s_i)} \end{aligned}$$

and

$$U_\theta = \frac{n}{\theta} + n \log \zeta - \sum_{i=1}^n \log x_i - \sum_{i=1}^n (\zeta/x_i)^\theta \log (\zeta/x_i) - (\lambda - 1) \sum_{i=1}^n \frac{z_i (1 - s_i)^{-1}}{\log (1 - s_i)},$$

where

$$\psi(\lambda) = \frac{d}{d\lambda} \{\log [\Gamma(\lambda)]\}, z_i = -\frac{\log(\zeta/x_i)}{(\zeta/x_i)^{-\theta}} s_i \text{ and } m_i = -\theta \zeta^{\theta-1} x_i^{-\theta} s_i.$$

Setting the nonlinear system of equations $U_\lambda = 0, U_\zeta = 0$ and $U_\theta = 0$ and solving them simultaneously yields the MLE $\hat{\eta} = (\hat{\lambda}, \hat{\zeta}, \hat{\theta})^\top$.

4 Simulation study

In this section, the performance of the maximum likelihood method is evaluated for estimating the ZBFR parameters using a Monte Carlo simulation study with 10,000 replications. The coverage probabilities (CPs), mean square errors (MSEs) and the bias of the parameter estimates, estimated average lengths (ALs) are calculated by means of R software. The quantities of these measures can be found in Cordeiro et al. [7]. We generate $N = 10,000$ samples of sizes $n = 50, 55, \dots, 1,000$ from the ZBFR distribution with $\lambda = 5, \zeta = 2, \theta = 12$.

Findings of the simulation experiments are graphically presented in Figure 2. From these plots it is easily observed that

- ✓ The estimated biases decrease when the sample size n increases,
- ✓ The estimated MSEs decay toward zero as n increases,
- ✓ The CPs are near 0.95 and approach the nominal value when the sample size n increases,
- ✓ The ALs decrease for all parameters when the sample size n increases.

These results reveal the consistency property of the MLEs.

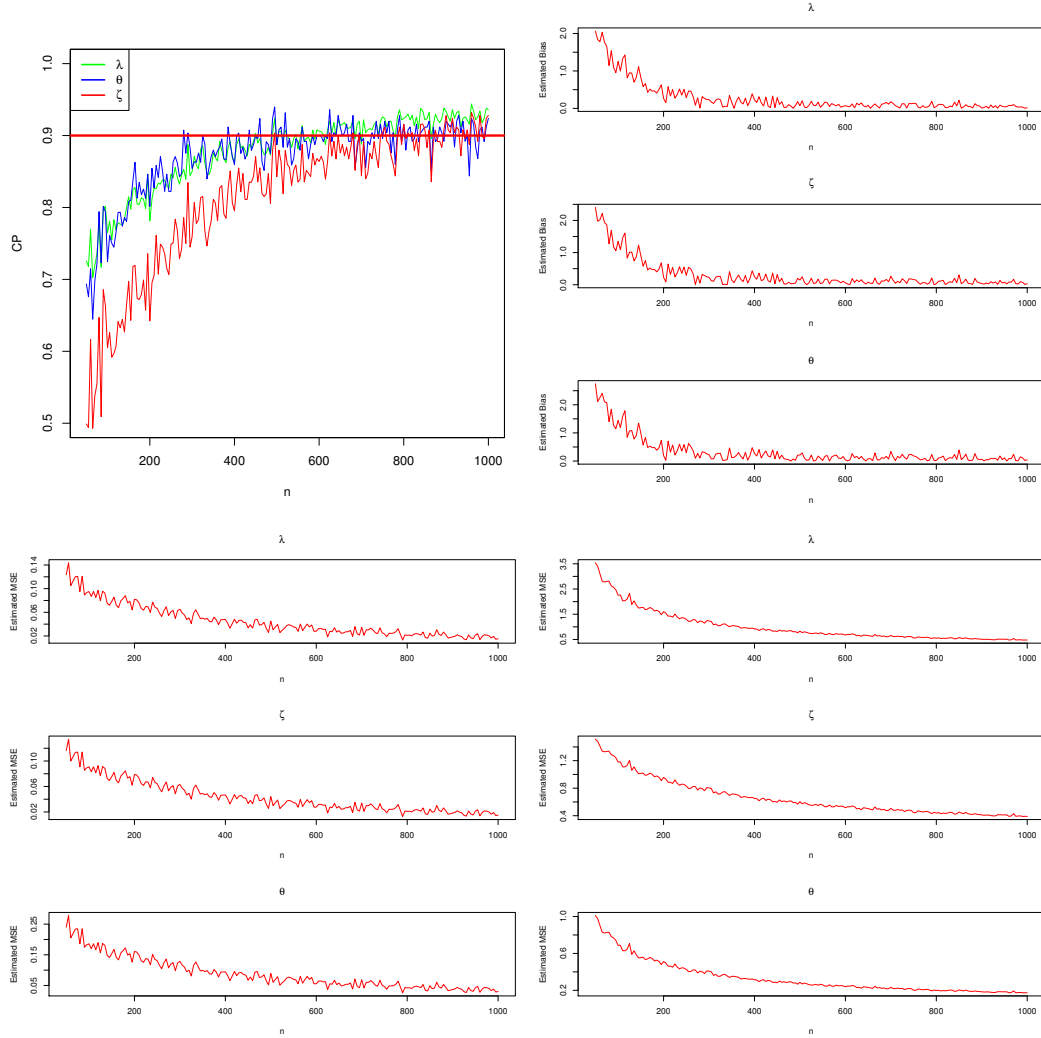


Figure 2: Estimated CPs, biases, MSEs and ALs for the selected parameter values.

5 Applications

Here, we consider two applications of real data modeling to illustrate the importance and potentiality of the ZB-Fr model. For these data, we compare the ZB-Fr distribution, with Fr, MO-Fr, Kw-Fr, B-Fr, MoKw-Fr and KwMO-Fr distributions given in Appendix.

The first data set comprises the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. This data set was previously used by Krishna et al. [14]. While the second data set represents the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia. These data was analyzed by Feigl and Zelen [12]

In order to select the best fitted model, we consider the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), consistent Akaike information criterion (CAIC). Generally, the smaller these statistics are, the better the fit. From our findings it is seen that in all the four cases considered here the proposed ZB-Fr turned out to be the best model in terms of different selection criteria. Moreover from the plots of estimated pdf against the observed histograms in Figure 3 and the estimated cdfs against empirical cdfs in Figure 3 reveals that the proposed distribution provides closest fit to all the data sets. It may be mentioned that the proposed three parameter distribution has even out performed the four and five parameter extensions considered here. The proposed new extension will therefore be a valued contribution to the existing set of extended Fr distribution.

Table 1: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC, values for the data set **I**

Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	AIC,BIC,CAIC,HQIC
Fr		54.199 (4.787)	1.414 (0.117)			795.30, 799.86, 795.47, 797.10
MO-Fr	61.775 (100.903)	14.204 (8.019)	2.481 (0.273)			789.42, 796.26, 789.77, 792.12
Kw-Fr		2.940 (0.138)	0.621 (0.205)	18.944 (5.339)	8.273 (7.862)	788.52, 797.64, 789.12, 792.14
B-Fr		2.845 (0.078)	0.215 (17.514)	69.748 (31.798)	43.846	789.78, 798.90, 790.38, 793.38
MoKw-Fr	(1.034) 0.007 (0.003)	(0.869) 2.666 (0.869)	(0.021) 0.449 (0.021)	(3.338) 22.880 (3.338)	1.376 (0.087)	794.18, 805.58, 795.09, 798.68
KwMo-Fr	7.642 (1.342)	12.704 (3.339)	1.402 (0.096)	2.434 (2.369)	2.035 (1.081)	792.22, 803.62, 793.13, 796.72
ZB-Fr	26.048 (0.597)	1.537 (0.008)	6.638 (0.007)			787.24, 794.08, 787.59, 789.94

Table 2: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC, values for the data set **II**

Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	AIC,BIC,CAIC,HQIC
Fr		7.866 (2.091)	0.694 (0.091)			315.98, 318.96, 316.38, 316.98
MO-Fr	8.085 (11.212)	1.904 (1.805)	0.987 (0.186)			315.38, 319.85, 316.21, 316.88
Kw-Fr		0.036 (0.075)	0.289 (0.131)	14.667 (9.022)	8.044 (10.920)	316.18, 322.14, 317.60, 318.18
B-Fr		0.070 (0.065)	0.117 (25.396)	45.081 (35.520)	31.414	316.16, 322.12, 317.58, 318.16
MoKw-Fr	(0.019) 2.314 (4.313)	0.049 (0.134)	0.320 (0.416)	11.705 (9.960)	6.397 (20.770)	318.00, 325.45, 320.22, 320.50
KwMo-Fr	3.887 (7.114)	0.009 (0.015)	0.386 (1.165)	9.797 (21.129)	4.716 (42.966)	318.62, 326.07, 320.84, 321.21
ZB-Fr	25.378 (2.747)	0.007 (0.005)	3.265 (0.0013)			310.50, 314.97, 311.33, 312.00

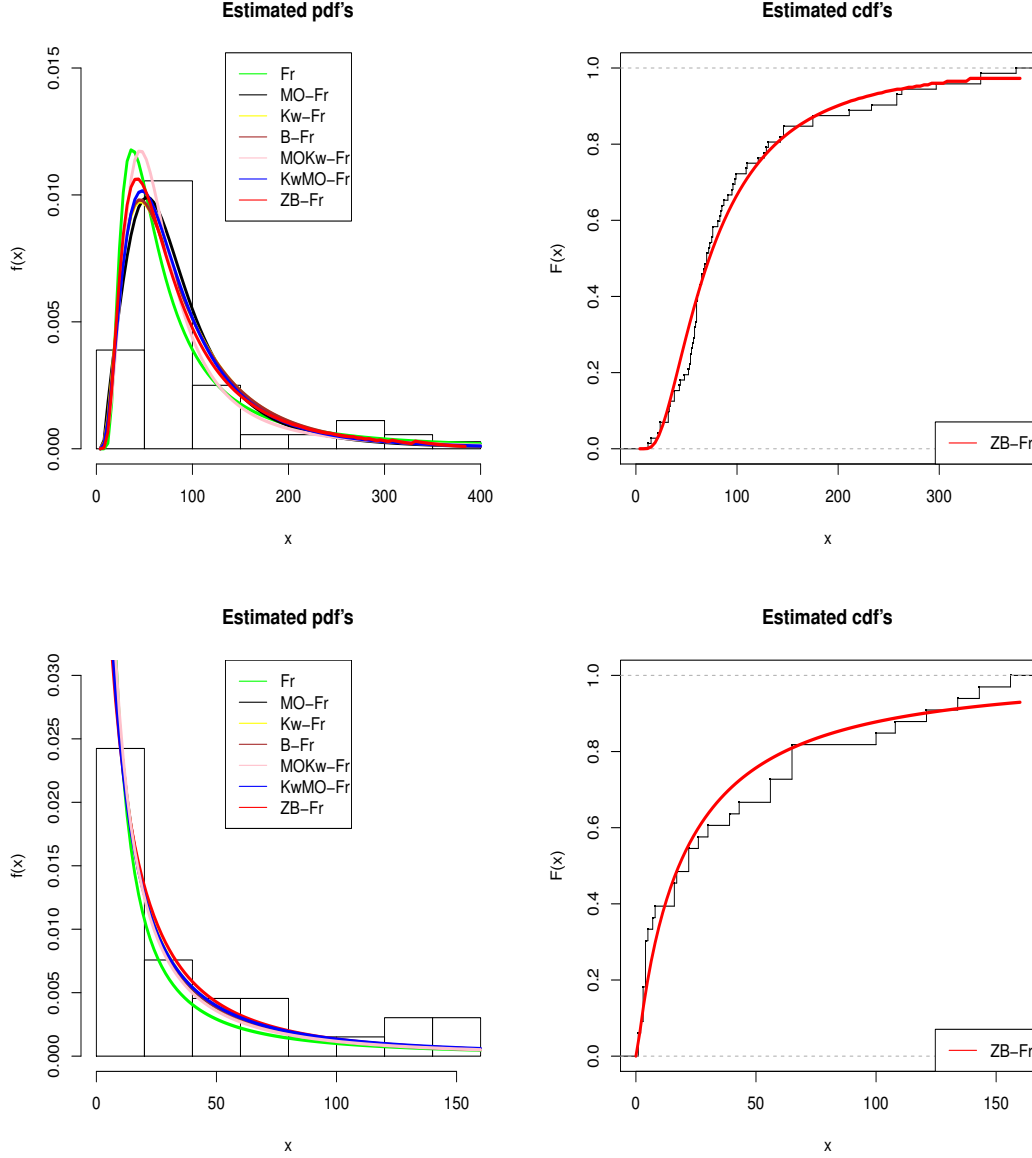


Figure 3: Plots of the observed histogram with estimated pdfs of all models(top right), observed and estimated of ZB-Fr (top left)for the data set I. For the data set II(bottom)

6 Conclusions

A new three parameter extreme value model which extends the Fréchet (Fr) distribution is defined and studied. Various structural properties of the proposed model including ordinary and incomplete moments, generating func-

tions, order statistics, residual and reversed residual life functions are investigated. The new density can be expressed as a linear mixture of the Fr density. Method of the maximum likelihood is used to estimate the model parameters. Various simulations are considered for different sample sizes to examine the performance of the maximum likelihood estimators for the model parameters. The new model is applied to two real data sets to prove empirically its flexibility. It can serve as an alternative extreme value model to other lifetime distributions in the existing literature for modeling the positive real data in many areas.

Appendix:

Here we provide the pdfs of all competing distributions used in the applications.

Frechet distribution:

$$f(x) = \theta \xi^\theta x^{-\theta-1} e^{-(\xi/x)^\theta}$$

MO-Fr distribution:

$$f(x) = \frac{\lambda \theta \xi^\theta x^{-\theta-1} e^{-(\xi/x)^\theta}}{[1 - \bar{\lambda} (1 - e^{-(\xi/x)^\theta})]^2}.$$

Kw-Fr distribution:

$$f(x) = a b \theta \xi^\theta x^{-\theta-1} e^{-(\xi/x)^\theta} [e^{-(\xi/x)^\theta}]^{a-1} [1 - \{e^{-(\xi/x)^\theta}\}^a]^{b-1}$$

B-Fr distribution :

$$f(x) = \frac{\theta \xi^\theta}{B(a, b)} x^{-\theta-1} [e^{-(\xi/x)^\theta}]^a [1 - e^{-(\xi/x)^\theta}]^{b-1}$$

MOKw-Fr distribution:

$$f(x) = \frac{\lambda a b \theta \xi^\theta x^{-\theta-1} e^{-(\xi/x)^\theta} [e^{-(\xi/x)^\theta}]^{a-1} [1 - \{e^{-(\xi/x)^\theta}\}^a]^{b-1}}{[1 - \bar{\lambda} [1 - \{e^{-(\xi/x)^\theta}\}^a]^b]^2}$$

KwMO-Fr distribution:

$$f(x) = \frac{\lambda a b \theta \xi^\theta x^{-\theta-1} e^{-(\xi/x)^\theta} [e^{-(\xi/x)^\theta}]^{a-1}}{1 - \bar{\lambda} [1 - e^{-(\xi/x)^\theta}]^{a+1}} \left[1 - \left\{ \frac{e^{-(\xi/x)^\theta}}{1 - \bar{\lambda} (1 - e^{-(\xi/x)^\theta})} \right\}^a \right]^{b-1}$$

The parameters of the above densities are all positive real numbers and the support is $x > 0$

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