

Local Fractional Homotopy Analysis Method for Solving Coupled Nonlinear System of Burger's Equations

Djelloul Ziane, Ahmed Bokhari, and Rachid Belgacem

Department of Physics, University of Hassiba Benbouali, Chlef, Algeria.
e-mail:d.ziane@univ-chlef.dz

Department of Mathematics, University of Hassiba Benbouali, Chlef, Algeria.
e-mail:a.bokhari@univ-chlef.dz

Department of Mathematics, University of Hassiba Benbouali, Chlef, Algeria.
e-mail:belgacemrachid02@yahoo.fr

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Abstract

The basic motivation of the present study is to apply the local fractional homotopy analysis method (HAM) to solve the coupled nonlinear system of Burger's equations with local fractional derivative. The derivatives operators are taken in the local fractional sense. The result obtained through the proposed system, proves the effectiveness of this method (LFHAM) in dealing with these types of problems, which is confirmed in relation to other works.

Keywords: *local fractional derivative operator, local fractional homotopy analysis method, local fractional nonlinear system of Burger's equations.*

1 Introduction

It is no secret to any researcher in the field of physics or engineering that systems of partial differential equation such as: $(1+1)$ -dimensional nonlinear

Boussinesq equation, coupled Hirota Satsuma KdV system, system of Maxwell equations, system of Schrodinger-Korteweg-De vries equations, system of coupled Burger's equations and other, can be used to modelled some phenomena such as, wave propagation and shallow water waves ([1], [2], [3]).

In addition to the physical and engineering importance, there is a second importance, lies in knowing the solutions of these equations or differential systems, whether exact or approximate. So we find a lot of researchers in this field are interested in developing several methods that enable us to solve this kind of equations or differential systems, among these methods we find, Adomian decomposition method (ADM) [4], homotopy perturbation method (HPM) [5], variational iteration method (VIM) [6], DJ-iteration method [7] and homotopy analysis method [8] and other, all this to have an ideal method to solve all kinds of differential equations as well as all kinds of differential systems.

Our work in this paper is based on the homotopy analysis method (HAM), which was developed in 1992 by Liao Shijun ([8]-[11]), and has been used extensively to solve partial or ordinary differential equations or differential systems of integer order or fractional order ([12]-[16]), where we will apply this method considered in the sense of the local fractional operator [23], to solve the coupled nonlinear system of Burger's equations with local fractional derivative.

The present paper has been organized as follows: In section 2 some basic definitions and properties of the local fractional derivative, local fractional integral and some important results. In section 3 we give an analysis of the local fractional homotopy analysis method (LFHAM). In section 4 we use the LFHAM to resolve the coupled nonlinear system of local fractional Burger's equations. Finally, the conclusion follows.

2 Basic definitions and theorems

In this section, we give some basic concepts about local fractional calculus, where we present the basic definitions and theories, including: Local fractional derivative, local fractional integral and some important results.

2.1 Local fractional derivative

Definition 1 *Setting $g(\tau) \in C_\sigma(\alpha, \beta)$, the local fractional derivative of $g(\tau)$ of order σ at $\tau = \tau_0$ is given by ([18], [19])*

$$g^{(\sigma)}(\tau) = \left. \frac{d^\sigma g}{d\tau^\sigma} \right|_{\tau=\tau_0} = \frac{\Delta^\sigma(g(\tau) - g(\tau_0))}{(\tau - \tau_0)^\sigma}, \quad (1)$$

where

$$\Delta^\sigma(g(\tau) - g(\tau_0)) \cong \Gamma(1 + \sigma) [(g(\tau) - g(\tau_0))]. \quad (2)$$

and $C_\sigma(\alpha, \beta)$ designates the class of functions called local fractional continuous on the interval (α, β) .

Local fractional derivative of high order is written in the form

$$g^{(m\sigma)}(\tau) = \overbrace{D_\tau^{(\sigma)} \cdots D_\tau^{(\sigma)}}^{m \text{ times}} g(\tau), \quad (3)$$

and local fractional partial derivative of high order

$$\frac{\partial^{m\sigma} g(\tau)}{\partial \tau^{m\sigma}} = \overbrace{\frac{\partial^\sigma}{\partial \tau^\sigma} \cdots \frac{\partial^\sigma}{\partial \tau^\sigma}}^{m \text{ times}} g(\tau). \quad (4)$$

2.2 Local fractional integral

Definition 2 The local fractional integral of $g(\tau)$ of order σ on the interval $[\alpha, \beta]$ is defined as ([18], [19])

$$\begin{aligned} {}_\alpha I_\beta^{(\sigma)} g(\tau) &= \frac{1}{\Gamma(1 + \sigma)} \int_\alpha^\beta g(\tau) (d\tau)^\sigma \\ &= \frac{1}{\Gamma(1 + \sigma)} \lim_{\Delta\tau_j \rightarrow 0} \sum_{j=0}^{N-1} f(\tau_j) (\Delta\tau_j)^\sigma, \end{aligned} \quad (5)$$

where $\Delta\tau_j = \tau_{j+1} - \tau_j$, with $\tau_0 = \alpha < \tau_1 < \cdots < \tau_N = \beta$.

2.3 Some important results

Definition 3 In fractal space, the Mittag Leffler function, sine function and cosine function are defined as ([18], [19])

$$E_\sigma(\tau^\sigma) = \sum_{m=0}^{+\infty} \frac{\tau^{m\sigma}}{\Gamma(1 + m\sigma)}, \quad 0 < \sigma \leq 1, \quad (6)$$

$$\sin_\sigma(\tau^\sigma) = \sum_{m=0}^{+\infty} (-1)^m \frac{\tau^{(2m+1)\sigma}}{\Gamma(1 + (2m+1)\sigma)}, \quad 0 < \sigma \leq 1, \quad (7)$$

$$\cos_\sigma(\tau^\sigma) = \sum_{m=0}^{+\infty} (-1)^m \frac{\tau^{2m\sigma}}{\Gamma(1 + 2m\sigma)}, \quad 0 < \sigma \leq 1, \quad (8)$$

The properties of local fractional derivative and integral of special functions are given by ([18], [19])

$$\frac{d^\sigma}{d\tau^\sigma} \frac{\tau^{m\sigma}}{\Gamma(1+m\sigma)} = \frac{\tau^{(m-1)\sigma}}{\Gamma(1+(m-1)\sigma)}, \quad (9)$$

$$\frac{d^\sigma}{d\tau^\sigma} E_\sigma(\tau^\sigma) = E_\sigma(\tau^\sigma), \quad (10)$$

$$\frac{d^\sigma}{d\tau^\sigma} \sin_\sigma(\tau^\sigma) = \cos_\sigma(\tau^\sigma), \quad (11)$$

$$\frac{d^\sigma}{d\tau^\sigma} \cos_\sigma(\tau^\sigma) = -\sin_\sigma(\tau^\sigma), \quad (12)$$

$${}_0I_\tau^{(\sigma)} \frac{\tau^{m\sigma}}{\Gamma(1+m\sigma)} = \frac{\tau^{(m+1)\sigma}}{\Gamma(1+(m+1)\sigma)}. \quad (13)$$

3 Local fractional homotopy analysis method

This method [23] serves for solving the nonlinear local fractional partial differential equation:

$$N[w(\varkappa, \tau)] = 0, \quad (14)$$

where N is the nonlinear operator, $\varkappa$ and τ denote the independent variables, and $w(\varkappa, \tau)$ denotes the local fractional unknown function. Using the same procedures described in the traditional homotopy analysis method [8], Maitama and Zhao [23] construct the so-called zero-order deformation equation:

$$(1-p)L[\phi(\varkappa, \tau; p) - w_0(\varkappa, \tau)] = phH(\varkappa, \tau)N[\phi(\varkappa, \tau; p)], \quad (15)$$

where $p \in [0, 1]$ is the embedding parameter, $h \neq 0$ is the non-zero auxiliary parameter, $H(\varkappa, \tau) \neq 0$ is an auxiliary function, $w_0(\varkappa, \tau)$ is an initial guess of $w(\varkappa, \tau)$ and $\phi(\varkappa, \tau; p)$ is an unknown function and $L_\sigma = \frac{\partial^\sigma}{\partial \tau^\sigma}$ is the linear local fractional operator with the property:

$$L_\sigma[\phi(\varkappa, \tau)] = 0, \quad \text{when } \phi(\varkappa, \tau) = 0. \quad (16)$$

For $p = 0$ and $p = 1$, Eq.(15) reduces to:

$$\phi(\varkappa, \tau; 0) = w_0(\varkappa, \tau) \text{ and } \phi(\varkappa, \tau; 1) = w(\varkappa, \tau), \quad (17)$$

respectively. Thus as p increases from 0 to 1, the solution $\phi(\mathcal{X}, \tau; p)$ varies from the initial guess $w_0(\mathcal{X}, \tau)$ to the solution $w(\mathcal{X}, \tau)$. Using the local fractional Taylor series with respect to p , the function $\phi(x, t; p)$ may be written as [18, 19]:

$$\phi(\mathcal{X}, \tau; p) = w_0(\mathcal{X}, \tau) + \sum_{m=1}^{+\infty} w_m(\mathcal{X}, \tau) p^m, \quad (18)$$

where

$$w_m(\mathcal{X}, \tau) = \left[\frac{1}{m!} \frac{\partial^m \phi(\mathcal{X}, \tau; p)}{\partial p^m} \right]_{p=0}. \quad (19)$$

With appropriate choices of the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function, the series (18) converges at $p = 1$, and turns into:

$$w(\mathcal{X}, \tau) = w_0(\mathcal{X}, \tau) + \sum_{m=1}^{+\infty} w_m(\mathcal{X}, \tau), \quad (20)$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao [9]. Define the vectors:

$$\vec{w}_n = \{w_0(\mathcal{X}, \tau), w_1(\mathcal{X}, \tau), \dots, w_n(\mathcal{X}, \tau)\}. \quad (21)$$

Differentiating the zeroth-order deformation Eq.(15) m -times with respect to p and dividing by $m!$, and then setting $p = 0$ gives the following m th-order deformation equation:

$$L_\sigma[w_m(\mathcal{X}, \tau) - \chi_m w_{m-1}(\mathcal{X}, \tau)] = h H(\mathcal{X}, \tau) \mathfrak{R}_m(\vec{w}_m, \mathcal{X}, \tau), \quad (22)$$

where

$$\mathfrak{R}_m(\vec{w}_m, \mathcal{X}, \tau) = \left[\frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(\mathcal{X}, \tau; p)]}{\partial p^{m-1}} \right]_{p=0}, \quad (23)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (24)$$

Applying the local fractional integral operator on both sides of Eq.(22), the result is:

$$\begin{aligned} w_m(\mathcal{X}, \tau) &= \chi_m w_{m-1}(\mathcal{X}, \tau) - \chi_m \sum_{k=0}^{m-1} w_{m-1}^{(k)}(\mathcal{X}, 0^+) \frac{\tau^{k\sigma}}{\Gamma(k\sigma + 1)} \\ &\quad + h_\beta I_\alpha^{(\sigma)} [H(\mathcal{X}, \tau) \mathfrak{R}_m(w_m, \mathcal{X}, \tau)]. \end{aligned} \quad (25)$$

It's easy to get the series solutions of $w_m(\varkappa, \tau)$ for $m \geq 1$ at m th-order deformation equation as follows:

$$w(\varkappa, \tau) = \sum_{m=0}^{+\infty} w_m(\varkappa, \tau). \quad (26)$$

For the convergence of the serie (26) you can see [23].

4 Applications

In this section we will apply the method described above which is attributed to Maitama and Zhao [23] to solve the following system when the derivatives operators are taken in the local fractional sense.

We consider the following nonlinear coupled Burger's equations with local fractional derivative:

$$\begin{cases} X_\tau^{(\sigma)} - X_{\varkappa\varkappa}^{(2\sigma)} - 2X X_\varkappa^{(\sigma)} + (XT)_\varkappa^{(\sigma)} = 0 \\ T_\tau^{(\sigma)} - T_{\varkappa\varkappa}^{(2\sigma)} - 2T T_\varkappa^{(\sigma)} + (XT)_\varkappa^{(\sigma)} = 0 \end{cases}, \quad 0 < \sigma \leq 1, \quad (27)$$

with the initial conditions:

$$X(\varkappa, 0) = \sin_\sigma(\varkappa^\sigma), \quad T(\varkappa, 0) = \sin_\sigma(\varkappa^\sigma). \quad (28)$$

Based on Eq.(27) and Eq.(28) and the procedure of the LFHAM, it is natural to choose $X_0(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma)$ and $T_0(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma)$ to be the initial guess.

We choose the linear operators as:

$$\begin{cases} L_\sigma [\phi(\varkappa, \tau; p)] = \phi_\tau^{(\sigma)}(\varkappa, \tau; p), \\ L_\sigma [\psi(\varkappa, \tau; p)] = \psi_\tau^{(\sigma)}(\varkappa, \tau; p), \end{cases} \quad (29)$$

with the property $L_\sigma[C] = 0$, where C is an integral constant.

We define the nonlinear operators as follows:

$$\begin{cases} N[\phi(x, t, p)] = \phi_\tau^{(\sigma)}(\varkappa, \tau; p) - \phi_{\varkappa\varkappa}^{(2\sigma)}(\varkappa, \tau; p) - 2\phi(\varkappa, \tau; p)\phi_\varkappa^{(\sigma)}(\varkappa, \tau; p) \\ \quad + (\phi(\varkappa, \tau; p)\psi(\varkappa, \tau; p))_\varkappa^{(\sigma)}, \\ N[\psi(x, t, p)] = \psi_\tau^{(\sigma)}(\varkappa, \tau; p) - \psi_{\varkappa\varkappa}^{(2\sigma)}(\varkappa, \tau; p) - 2\psi(\varkappa, \tau; p)\psi_\varkappa^{(\sigma)}(\varkappa, \tau; p) \\ \quad + (\phi(\varkappa, \tau; p)\psi(\varkappa, \tau; p))_\varkappa^{(\sigma)}. \end{cases} \quad (30)$$

We construct the zero-order deformation equations:

$$\begin{cases} (1-p)L[\phi(\mathfrak{x}, \tau; p) - X_0(\mathfrak{x}, \tau)] = phH(\mathfrak{x}, \tau)N[\phi(\mathfrak{x}, \tau; p)], \\ (1-p)L[\psi(\mathfrak{x}, \tau; p) - T_0(\mathfrak{x}, \tau)] = phH(\mathfrak{x}, \tau)N[\psi(\mathfrak{x}, \tau; p)]. \end{cases} \quad (31)$$

Obviously, when $p = 0$ and $p = 1$, we get:

$$\begin{cases} \phi(\mathfrak{x}, \tau; 0) = X_0(\mathfrak{x}, \tau), \\ \psi(\mathfrak{x}, \tau; 0) = T_0(\mathfrak{x}, \tau), \end{cases} \quad \text{and} \quad \begin{cases} \phi(\mathfrak{x}, \tau; 1) = X(\mathfrak{x}, \tau), \\ \psi(\mathfrak{x}, \tau; 1) = T(\mathfrak{x}, \tau). \end{cases} \quad (32)$$

Then the m th-order deformation equations are defined as follows:

$$\begin{cases} L_\sigma[X_m(\mathfrak{x}, \tau) - \chi_m X_{m-1}(\mathfrak{x}, \tau)] = hH(\mathfrak{x}, \tau) \mathfrak{R}_m(\vec{X}_{m-1}, \mathfrak{x}, \tau), \\ L_\sigma[T_m(\mathfrak{x}, \tau) - \chi_m T_{m-1}(\mathfrak{x}, \tau)] = hH(\mathfrak{x}, \tau) \mathfrak{R}_m(\vec{T}_{m-1}, \mathfrak{x}, \tau), \end{cases} \quad (33)$$

where

$$\begin{cases} \mathfrak{R}_m(\vec{X}_{m-1}, \mathfrak{x}, \tau) = (X_{m-1})_\tau^{(\sigma)} - (X_{m-1})_{\mathfrak{x}\mathfrak{x}}^{(2\sigma)} - 2X_{m-1}(X_{m-1})_\mathfrak{x}^{(\sigma)} \\ \quad + (X_{m-1}T_{m-1})_\mathfrak{x}^{(\sigma)}, \\ \mathfrak{R}_m(\vec{T}_{m-1}, \mathfrak{x}, \tau) = (T_{m-1})_\tau^{(\sigma)} - (T_{m-1})_{\mathfrak{x}\mathfrak{x}}^{(2\sigma)} - 2T_{m-1}(T_{m-1})_\mathfrak{x}^{(\sigma)} \\ \quad + (X_{m-1}T_{m-1})_\mathfrak{x}^{(\sigma)}, \end{cases} \quad (34)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (35)$$

Setting $H(\mathfrak{x}, \tau) = 1$ and applying the local fractional integral on the m^{th} -order deformation Eqs.(33), we get:

$$\begin{aligned} X_m &= (\chi_m + h) [X_{m-1} - (1 - \chi_m) X(\mathfrak{x}, 0)] \\ &\quad - h_0 I_\tau^{(\sigma)} \left[(X_{m-1})_{\mathfrak{x}\mathfrak{x}}^{(2\sigma)} + 2 \sum_{i=0}^{m-1} X_i (X_{m-1-i})_\mathfrak{x}^{(\sigma)} - \sum_{i=0}^{m-1} (X_i T_{m-1-i})_\mathfrak{x}^{(\sigma)} \right], \\ T_m &= (\chi_m + h) [T_{m-1} - (1 - \chi_m) T(\mathfrak{x}, 0)] \\ &\quad - h_0 I_\tau^{(\sigma)} \left[(T_{m-1})_{\mathfrak{x}\mathfrak{x}}^{(2\sigma)} + 2 \sum_{i=0}^{m-1} T_i (T_{m-1-i})_\mathfrak{x}^{(\sigma)} - \sum_{i=0}^{m-1} (X_i T_{m-1-i})_\mathfrak{x}^{(\sigma)} \right]. \end{aligned} \quad (36)$$

Based to the (35) and (36), we obtain the first formulas of the three terms

as follows:

$$\begin{cases}
 X_1(\varkappa, \tau) = -h_0 I_\tau^{(\sigma)} \left[(X_0)_{\varkappa\varkappa}^{(2\sigma)} + A_0(X, T) \right], \\
 T_1(\varkappa, \tau) = -h_0 I_\tau^{(\sigma)} \left[(T_0)_{\varkappa\varkappa}^{(2\sigma)} + B_0(X, T) \right], \\
 \begin{cases}
 X_2(\varkappa, \tau) = (1+h)X_1 - h_0 I_\tau^{(\sigma)} \left[(X_1)_{\varkappa\varkappa}^{(2\sigma)} + A_1(X, T) \right], \\
 T_2(\varkappa, \tau) = (1+h)T_1 - h_0 I_\tau^{(\sigma)} \left[(T_1)_{\varkappa\varkappa}^{(2\sigma)} + B_1(X, T) \right], \\
 X_3(\varkappa, \tau) = (1+h)X_2 - h_0 I_\tau^{(\sigma)} \left[(X_2)_{\varkappa\varkappa}^{(2\sigma)} + A_2(X, T) \right], \\
 T_3(\varkappa, \tau) = (1+h)T_2 - h_0 I_\tau^{(\sigma)} \left[(T_2)_{\varkappa\varkappa}^{(2\sigma)} + B_2(X, T) \right], \\
 \vdots
 \end{cases}
 \end{cases} \quad (37)$$

where

$$\begin{aligned}
 A_0(X, T) &= 2X_0(X_0)_{\varkappa}^{(\sigma)} - (X_0 T_0)_{\varkappa}^{(\sigma)}, \\
 B_0(X, T) &= 2T_0(T_0)_{\varkappa}^{(\sigma)} - (X_0 T_0)_{\varkappa}^{(\sigma)}, \\
 A_1(X, T) &= 2 \left(X_0(X_1)_{\varkappa}^{(\sigma)} + X_1(X_0)_{\varkappa}^{(\sigma)} \right) - (X_0 T_1 + X_1 T_0)_{\varkappa}^{(\sigma)}, \\
 B_1(X, T) &= 2 \left(T_0(T_1)_{\varkappa}^{(\sigma)} + T_1(T_0)_{\varkappa}^{(\sigma)} \right) - (X_0 T_1 + X_1 T_0)_{\varkappa}^{(\sigma)}, \\
 A_2(X, T) &= 2 \left[X_0(X_2)_{\varkappa}^{(\sigma)} + X_1(X_1)_{\varkappa}^{(\sigma)} + X_2(X_0)_{\varkappa}^{(\sigma)} \right] - (X_0 T_2 + X_1 T_1 + X_2 T_0)_{\varkappa}^{(\sigma)}, \\
 B_2(X, T) &= 2 \left[T_0(T_2)_{\varkappa}^{(\sigma)} + T_1(T_1)_{\varkappa}^{(\sigma)} + T_2(T_0)_{\varkappa}^{(\sigma)} \right] - (X_0 T_2 + X_1 T_1 + X_2 T_0)_{\varkappa}^{(\sigma)}, \\
 &\quad (38)
 \end{aligned}$$

and so on. According to the initial guess presented above and the formulas (37) and (38), the first terms of local fractional homotopy analysis method of the system (27), are given by:

$$\begin{cases}
 X_0(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma), \\
 T_0(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma), \\
 \begin{cases}
 X_1(\varkappa, \tau) = h \sin_\sigma(\varkappa^\sigma) \frac{\tau^\sigma}{\Gamma(1+\sigma)}, \\
 T_1(\varkappa, \tau) = h \sin_\sigma(\varkappa^\sigma) \frac{\tau^\sigma}{\Gamma(1+\sigma)}, \\
 \begin{cases}
 X_2(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[(h+h^2) \frac{\tau^\sigma}{\Gamma(1+\sigma)} + h^2 \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} \right], \\
 T_2(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[(h+h^2) \frac{\tau^\sigma}{\Gamma(1+\sigma)} + h^2 \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} \right], \\
 \begin{cases}
 T_3(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[h(1+h)^2 \frac{\tau^\sigma}{\Gamma(1+\sigma)} + 2(h^2+h^3) \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} + h^3 \frac{\tau^{3\sigma}}{\Gamma(1+3\sigma)} \right], \\
 T_3(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[h(1+h)^2 \frac{\tau^\sigma}{\Gamma(1+\sigma)} + 2(h^2+h^3) \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} + h^3 \frac{\tau^{3\sigma}}{\Gamma(1+3\sigma)} \right], \\
 \vdots
 \end{cases}
 \end{cases}
 \end{cases} \quad (39)$$

and so on. The other components of the LFHAM can be determined in a similar way. Finally, the approximate coupled solution (X, T) of the system (27) in a series form, is given by:

$$\begin{cases} X(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[1 + h(3 + 3h + h^2) \frac{\tau^\sigma}{\Gamma(1+\sigma)} + h^2(3 + 2h) \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} \right. \\ \quad \left. + h^3 \frac{\tau^{3\sigma}}{\Gamma(1+3\sigma)} + \dots \right], \\ T(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[1 + h(3 + 3h + h^2) \frac{\tau^\sigma}{\Gamma(1+\sigma)} + h^2(3 + 2h) \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} \right. \\ \quad \left. + h^3 \frac{\tau^{3\sigma}}{\Gamma(1+3\sigma)} + \dots \right]. \end{cases} \quad (40)$$

Substituting $h = -1$ in (39), we obtain:

$$\begin{cases} X(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[1 - \frac{\tau^\sigma}{\Gamma(1+\sigma)} + \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} - \frac{\tau^{3\sigma}}{\Gamma(1+3\sigma)} + \dots \right], \\ T(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) \left[1 - \frac{\tau^\sigma}{\Gamma(1+\sigma)} + \frac{\tau^{2\sigma}}{\Gamma(1+2\sigma)} - \frac{\tau^{3\sigma}}{\Gamma(1+3\sigma)} + \dots \right]. \end{cases} \quad (41)$$

And in the closed form, the coupled solution (X, T) of the system (27), is given by:

$$\begin{cases} X(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) E_\sigma(-\tau^\sigma), \\ T(\varkappa, \tau) = \sin_\sigma(\varkappa^\sigma) E_\sigma(-\tau^\sigma). \end{cases} \quad (42)$$

Note that, in the case $\sigma = 1$, we obtain the same solution obtained in [21] by homotopy perturbation method, and in [22] by natural decomposition method.

5 Conclusion

The application of the local fractional homotopy analysis method (LFHAM) for solving the coupled nonlinear system of Burger's equations within local fractional derivative, gave us a powerful and efficient algorithm that enabled us to solve this proposed system with local fractional derivative. This algorithm provides the solution in a series form that converges rapidly to the exact solution, as shown by the results obtained through to the example suggested in this paper. Therefore, it can be said that this method (LFHAM) can be applied to solve other nonlinear systems with local fractional derivative.

6 Open problem

In this paper, we used the LFHAM to resolve the coupled nonlinear system of local fractional Burgers equations. The open problem here, we can that this method (LFHAM) can be applied to solve other nonlinear systems with local fractional derivative. We can also combine this method and the new integral transform, namely, Shehu transform ([23],[24],[25]), which generalize both the Sumudu and Laplace integral transforms.

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