

Transcendence criterion in number system with negative base

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Abstract

Let p be an integer. We prove that if $d_{-p}(x) = (a_i)_{i \geq 1}$ is a stammering sequence, then x is rational or transcendent. This result provides, in a natural way, a family of transcendental numbers.

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1 Introduction

The purpose of this paper is to give a criterion of transcendence when β is an integer. The paper is organized as follows: In section 2, we recall the definition of $(-\beta)$ -expansion in the negative base. In section 3, we prove the main result. Representations of real numbers with an arbitrary base $\beta > 1$, called β -expansions, were introduced by Rényi [8] takes a very special place. Since then, researchers were looking at the links between symbolic dynamics

and number theory of β -expansions, when $\beta > 1$ is an algebraic number or more generally a real number.

Let $\beta > 1$ be a real number. The β -expansion of a real number $x \in [0, 1[$, noted by $d_\beta(x)$, and defined as follows:

$$d_\beta(x) = 0.a_1a_2\cdots$$

where the sequence $(a_i)_{i \geq 1}$ produced by the β -transformation $T_\beta : x \rightarrow \beta x \pmod{1}$ as follows:

$$\forall i \geq 1, x_i = [\beta T_\beta^{i-1}(x)], \text{ and thus } x = \sum_{i \geq 1} \frac{x_i}{\beta^i}.$$

Note that if $\beta = b \in \mathbb{N}$, then $d_b(x)$ coincides with the ordinary b -ary expansion of x , but only the β -expansion of 1 differs, in which case $d_\beta(1) = 0.(\beta - 1)(\beta - 1) \cdots$ and for $x < 1$, the digits a_i of $d_b(x)$ lie in the set $\{0, 1, \dots, b - 1\}$. If β is not an integer, then the digits a_i lie in the set $A = \{0, 1, \dots, [\beta]\}$. So the β -expansion generalize standard representations in an integer base to a real base; indeed this expansion of a real number $x \in [0, 1[$ can equivalently be obtained by the last algorithm.

In [7], Parry has provided a criterion which decide whether an infinite word in $A^\mathbb{N}$ is or not a β -expansion of some real number x . The criteria are formulated using the so-called infinite expansion of 1, denoted by $d_\beta^*(1)$ defined by

$$d_\beta^*(1) = \begin{cases} d_\beta(1), & \text{if } d_\beta(1) \text{ is infinite;} \\ (t_1 \dots t_{m-1}, t_m - 1)^\infty, & \text{if } d_\beta(1) = 0.t_1 \dots t_m. \end{cases}$$

In the following, we say that $(x_i)_{i \in \mathbb{N}^*}$ is admissible (with respect to β) if it satisfies the Parry condition:

$$\forall n \in \mathbb{N}^*, \sigma^n((x_i)_{i \in \mathbb{N}^*}) <_{lex} d_\beta^*(1),$$

where $\sigma((x_i)_{i \in \mathbb{N}^*}) = (x_{i+1})_{i \in \mathbb{N}^*}$.

Moreover, it seems to be very difficult to describe the β -expansion of an algebraic number when this expansion is neither finite nor eventually periodic. In this context, Blanchard [2] proposed a classification of the properties of the (finite or infinite) word given by $d_\beta(1)$. The occurrences of consecutive 0's in $d_\beta(1)$ play a crucial role in Blanchard's classification of algebraic numbers β . This motivates the following problem first investigates in [14] by Verger-Gaugry. He studies the strings of zeros ("gaps") in the Rényi β -expansion $d_\beta(1)$. Using a version of Liouville's inequality which extends Mahler's [6] and Guting's approximation theorems [4], the strings of zeros in $d_\beta(1)$ are shown to exhibit a "gappiness" asymptotically bounded above by $\frac{\log(M(\beta))}{\log \beta}$, where $M(\beta)$ is the Mahler measure of β defined by the product

$$M(\beta) = |\alpha_d| \prod_{i=0}^{d-1} \max\{1, |\beta_i|\},$$

where $\beta = \beta_0, \beta_1, \dots, \beta_{d-1}$ are the complex conjugates of β and a_d is the leading coefficient of the minimal defining polynomial of β over the integers.

Let A be a finite set. The length of a word W on the alphabet A , that is, the number of letters composing W , is denoted by $|W|$. If a is a letter and W a finite word, then $|W|_a$ denotes the number of occurrences of the letter a in W . For any positive integer k , we write W^k for the word $W \cdots W$ (the concatenation of the word W repeated k times).

More generally, for any positive real number x , W^x denotes the word $W^{\lfloor x \rfloor} W'$ where W' is the prefix of W of length $\lceil (x - \lfloor x \rfloor)|W| \rceil$. Here, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor and ceiling functions, respectively.

We say that an infinite sequence $a = (a_i)_{i \geq 1}$ over a finite alphabet A satisfies Condition $(*)$ if there exist two sequences of finite words $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ over A and $\omega > 1$ such that:

i) For any $n \geq 1$, the word $U_n V_n^\omega$ is a prefix of $\mathbf{a}$.

ii) $(|U_n|/|V_n|)_{n \geq 1}$ is bounded. (*)

iii) $(|V_n|)_{n \geq 1}$ is increasing.

Recently, in analogy to the Rényi expansion of positive numbers using the transformation T_β of the interval $[0, 1)$, Ito and Sadahiro [5] have suggested studying the $(-\beta)$ -expansion of all real numbers in base $-\beta$, where $\beta > 1$, i.e. in the form

$$x = \sum_{i=-\infty}^k x_i (-\beta)^i,$$

where $x_i \in \{0, 1, \dots, \lfloor \beta \rfloor\}$ using the new transformation $T_{-\beta}$ of the interval $I_\beta = [l_\beta, r_\beta) = [\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$. They have provided a criterion for admissibility of digit strings as $(-\beta)$ -expansions of a real number x noted $d_{-\beta}(x)$, indeed they have shown some proprieties of the dynamical system connected to $(-\beta)$ -numeration.

2 Preliminaries

The representation of real numbers in such a system is defined using the transformation $T_{-\beta} : [l_\beta, r_\beta) \mapsto [l_\beta, r_\beta)$, where $l_\beta = \frac{-\beta}{\beta+1}$, $r_\beta = \frac{1}{\beta+1} = 1 + l_\beta$. The transformation $T_{-\beta}$ is given by the prescription $T_{-\beta}(x) := -\beta x - \lfloor \beta x - l_\beta \rfloor$.

Every real $x \in I_\beta = [l_\beta, r_\beta)$ is written in the form $x = \sum_{i=0}^{+\infty} x_i (-\beta)^i$ is said to be a greedy expansion, if $x_i = \lfloor -\beta T_{-\beta}^{i-1}(x) - l_\beta \rfloor$ for $i = 1, 2, 3, \dots$

The above expression is called the $(-\beta)$ -expansion of x . It can also be written as the infinite word $d_{-\beta}(x) = x_1x_2x_3\dots$. We can easily derive from the definition of the transformation $T_{-\beta}$ that the digits x_i take values in the set $A_\beta = \{0, 1, \dots, [\beta]\}$ for $i = 1, 2, \dots$. The advantage of this numeration system with negative base stems from the fact that both positive and negative real numbers can be represented with non-negative digits, without the necessity to mark the sign of the number. Thereby, in [5] it is suggested to find the expansion of a number $x \notin [l_\beta, r_\beta)$ by dividing it by a suitable power of $(-\beta)$ so that $y := (-\beta)^k x \in [l_\beta, r_\beta)$, finding the expansion of y and multiplying it back by $(-\beta)^k$. The expression for x provided by such procedure, however, depends on chosen k , so the prescription must be modified, in order to give a unique $(-\beta)$ -expansion for every real x . In this case, the ordering on the set of infinite words over the set A_β which could correspond to the ordering of real numbers is the so-called alternate ordering: We say that $x_1x_2x_3\dots <_{alt} y_1y_2y_3\dots$ if for the minimal index j such that $x_j \neq y_j$, it holds that $x_j(-1)^j < y_j(-1)^j$. Nevertheless, in [5] Ito and Sadahiro have provided a criterion to decide a string $x_1x_2x_3\dots$ of positive integers is said to be $(-\beta)$ -admissible if and only if every suffix of it satisfies the following condition:

$$\forall i \in \mathbb{N}^*, d_{-\beta}(l_\beta) \leq_{alt} (x_i x_{i+1} x_{i+2} \dots) <_{alt} d_{-\beta}^*(r_\beta).$$

where

$$d_{-\beta}^*(r_\beta) = \begin{cases} (0b_1b_2\dots b_{q-1}b_q - 1)^\omega, & \text{if } d_{-\beta}(l_\beta) = (b_1b_2\dots b_q)^\omega \text{ and } q \text{ odd;} \\ d_{-\beta}(r_\beta), & \text{otherwise.} \end{cases}$$

Some arithmetical properties of $(-\beta)$ -numeration systems are studied in [10] and [9]. Representations of numbers in the numeration system from the point of view of dynamical systems have been studied by A. Chiara Lai and C. Frougny in [3]. They have proven this theorem:

Theorem 2.1 [3] *If β is a Pisot number, Then $d_{-\beta}(x)$ is eventually periodic for any $x \in I_\beta \cap \mathbb{Q}(\beta)$.*

3 Main results

Theorem 3.1 *Let p be an integer, if $d_{-p}(x)$ verifies (*) and $d_{-p}(x)$ is not periodic then x is transcendental.*

In order to prove this result, we need the famous p -adic subspace theorem:

Theorem 3.2 (Schlickewei, 1977) [11] *Let $n \geq 2$, $\varepsilon > 0$, and let $p_1, \dots, p_s$ be distinct prime numbers. Further, let $L_{1,\infty}, \dots, L_{r,\infty}$ be linearly independent linear forms in $X_1 \dots X_n$ with algebraic coefficients in $\mathbb{C}$ that are algebraic over*

$\mathbb{Q}$, and for $j = 1, \dots, s$, let $L_{1,p_j}, \dots, L_{n,p_j}$ be linearly independant linear forms in $X_1 \dots X_n$ with coefficients in $\overline{\mathbb{Q}}_{p_j}$ that are algebraic over $\mathbb{Q}$. Consider the inequality

$$|L_{1\infty}(x) \dots L_{n\infty}(x)| \cdot \prod_{j=1}^s |L_{1,p_j}(x) \dots L_{n,p_j}(x)|_{p_j} \leq \|x\|^{-\varepsilon} \text{ in } x \in \mathbb{Z}^n \quad (1)$$

where for $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, $\|x\| = \max(|x_1|, \dots, |x_n|)$.

Then there are a finite number of proper linear subspaces $T_1, \dots, T_t$ of $\mathbb{Q}^n$ such that all solutions of (1) lie in $T_1 \cup \dots \cup T_t$.

Proof of Theorem 3.1. Let p be an integer such that $p = p_1^{\gamma_1} \dots p_k^{\gamma_k} \dots$ its prime factor decomposition, and let $\alpha \in [\frac{-p}{p+1}, \frac{1}{p+1}[$,

$$\alpha = \sum_{i \geq 1} \frac{a_i}{(-p)^i}, \quad \text{where } |a_i| < p$$

$$\begin{aligned} \forall n, d_\beta(\alpha) &= (a_i)_{i \geq 0} \\ &= U_n V_n^\omega. \end{aligned}$$

Let $\alpha'_n = U_n \overline{V_n}$ a periodic sequence and let $r_n = |U_n|$, $s_n = |V_n|$.

$$\begin{aligned} \alpha'_n &= \frac{a_0}{(-p)} + \frac{a_1}{(-p)^2} + \dots + \frac{a_{r_n}}{(-p)^{r_n}} + \frac{a_{r_n+1}}{(-p)^{r_n+1}} + \dots + \frac{a_{r_n+s_n}}{(-p)^{r_n+s_n}} + \dots \\ &+ \frac{a_{r_n+2s_n}}{(-p)^{r_n+2s_n}} + \dots \\ &= \frac{a_0}{(-p)} + \frac{a_1}{(-p)^2} + \dots + \frac{a_{r_n}}{(-p)^{r_n}} + \frac{a_{r_n+1}}{(-p)^{r_n+1}} + \dots + \frac{a_{r_n+s_n}}{(-p)^{r_n+s_n}} \\ &+ \frac{1}{(-p)^{s_n}} (\alpha'_n - \frac{a_0}{(-p)} - \dots - \frac{a_{r_n}}{(-p)^{r_n}}) \end{aligned}$$

Subtracting by $\frac{\alpha'_n}{(-p)^{s_n}}$, we obtain:

$$\begin{aligned} \alpha'_n (1 - \frac{1}{(-p)^{s_n}}) &= \frac{a_0}{(-p)} + \frac{a_1}{(-p)^2} + \dots + \frac{a_{r_n}}{(-p)^{r_n}} + \frac{a_{r_n+1}}{(-p)^{r_n+1}} + \dots + \frac{a_{r_n+s_n}}{(-p)^{r_n+s_n}} \\ &- \frac{a_0}{(-p)^{s_n+1}} - \dots - \frac{a_{r_n}}{(-p)^{r_n+s_n}} \\ &= \sum_{i=1}^{r_n+s_n} \frac{b_i}{(-p)^i}, \text{ where } b_i \in [-(p-1), p-1[\end{aligned}$$

$$\alpha'_n\left(\frac{(-p)^{s_n}-1}{(-p)^{s_n}}\right) = \sum_{i=1}^{r_n+s_n} \frac{b_i}{(-p)^i}$$

This implies that,

$$\begin{aligned} \alpha'_n((-p)^{s_n}-1) &= \sum_{i=1}^{r_n} \frac{b_i}{(-p)^i} \\ &= \frac{H_n(p)}{(-p)^{r_n}} \end{aligned}$$

We obtain finally that,

$$\alpha'_n = \frac{H_n(p)}{(-p)^{r_n}((-p)^{s_n}-1)} \quad (2)$$

We suppose now that α is algebraic. Let $X_n = ((-p)^{r_n+s_n}, -(-p)^{r_n}, -H_n(p)) \in \mathbb{Z}^3$ where $X = (x, y, z) \in \mathbb{Z}^3$.

Further, let $L_{1,\infty}$, $L_{2,\infty}$ and $L_{3,\infty}$ be linearly independant linear forms defined as follows:

$$\begin{aligned} L_{1,\infty} &= \alpha x + \alpha y + z \\ L_{2,\infty} &= x \\ L_{3,\infty} &= y \end{aligned}$$

First, we begin by calculating $|L_{1,\infty}(X_n)|$, $|L_{2,\infty}(X_n)|$ and $|L_{3,\infty}(X_n)|$.

$$\begin{aligned} |L_{1,\infty}(X_n)| &= |\alpha(-p)^{r_n+s_n} - \alpha(-p)^{r_n} - H_n(p)| \\ &= |\alpha((-p)^{r_n+s_n} - (-p)^{r_n}) - H_n(p)|, \end{aligned}$$

By using (2) we obtain,

$$|L_{1,\infty}(X_n)| = |\alpha((-p)^{r_n+s_n} - (-p)^{r_n}) - \alpha'_n(-p)^{r_n}((-p)^{s_n}-1)|$$

On the other hand we have,

$$\alpha - \alpha'_n = -\frac{a_{r_n+\omega s_n+1}}{(-p)^{r_n+\omega s_n+1}} + \dots$$

This gives that,

$$|\alpha - \alpha'_n| \leq \frac{c}{p^{r_n+\omega s_n+1}} \quad (3)$$

Therefore,

$$|L_{1,\infty}(X_n)| = |((-p)^{r_n+s_n} - (-p)^{r_n})(\alpha - \alpha'_n)|$$

Using (3), we obtain:

$$|L_{1,\infty}(X_n)| \leq \frac{1}{p^{s_n(\omega-1)+1}}$$

Moreover we have,

$$\begin{aligned} |L_{2,\infty}(X_n)| &= p^{r_n+s_n} \\ |L_{3,\infty}(X_n)| &= p^{r_n} \end{aligned}$$

Now, let $L_{1,p}, L_{2,p}, L_{3,p}$ be three linear forms in X_1, X_2, X_3 with algebraic coefficients in $\mathbb{Q}$, defined as follows for all $1 \leq i \leq h$:

$$\begin{aligned} L_{1,p_i}(X) &= x \\ L_{2,p_i}(X) &= y \\ L_{3,p_i}(X) &= z \text{ where } X = (x, y, z) \in \mathbb{Z}^3 \end{aligned}$$

Thus we have,

$$\begin{aligned} \prod_{i=1}^h |L_{1,p_i}(X_n)|_{p_i} &= p^{-(r_n+s_n)} \\ \prod_{i=1}^h |L_{2,p_i}(X_n)|_{p_i} &= p^{-r_n} \\ \prod_{i=1}^h |L_{3,p_i}(X_n)|_{p_i} &= H_n(p) \leq 1. \end{aligned}$$

Thanks to condition (*), there exists $\gamma > 0$ such that $\frac{|U_n|}{|V_n|} = \gamma$, thus $\frac{r_n}{s_n} < \gamma$ so $r_n + s_n < (\gamma + 1)s_n$, we obtain then that $s_n > \frac{r_n+s_n}{\gamma+1}$.

$$|L_{1,\infty}(X_n)L_{2,\infty}(X_n)L_{3,\infty}(X_n)| \prod_{i=1}^h |L_{1,p_i}(X_n)L_{2,p_i}(X_n)L_{3,p_i}(X_n)|_{p_i} \leq \frac{1}{p^{(r_n+s_n)(\frac{\omega-1}{\gamma+1})}}$$

This means that,

$$|L_{1,\infty}(X_n)L_{2,\infty}(X_n)L_{3,\infty}(X_n)| \prod_{i=1}^h |L_{1,p_i}(X_n)L_{2,p_i}(X_n)L_{3,p_i}(X_n)|_{p_i} \leq \frac{1}{\|X_n\|^\varepsilon} \quad (4)$$

where $\|X_n\| = \frac{1}{p^{(r_n+s_n)}}$ and $\varepsilon = \frac{\omega-1}{\gamma+1}$.

Thanks to Theorem 3.2, we conclude that there exists $(a, b, c) \in \mathbb{Q}^3$ and an infinite number of X_n such that $a((-p)^{(r_n+s_n)}) - b(-p)^{r_n} - cH_n(p) = 0$.

Dividing by $(-p)^{(r_n+s_n)}$ yields,

$$a - b(-p)^{-(s_n)} - c\alpha'_n(1 - (-p)^{-(s_n)}) = 0 \quad (5)$$

Since s_n is increasing and α'_n tend to α when n tend to $+\infty$, then by tending n to $+\infty$ in 5, we obtain $a - c\alpha = 0$. We conclude finally that α is rational. $\square$

Example 3.1 For any finite bloc $W = a_1 \dots a_s$, we define $W^* = a_1 \dots a_{s-1}$ and $W' = W^{\frac{1}{2}}$.

Now, let the sequences of bloc $(W_n)_{n \geq 0}$ defined by $W_0 = 10$, $W_1 = 1011$ and $W_n = W_{n-1}W_{n-1}^*W_{n-1}'$ for all $n \geq 2$, let $W = \lim_{n \rightarrow +\infty} W_n = (a_i)_{i \geq 0}$ (the limit exists because W_n begins by W_{n-1}).

We can easily see that W is not periodic. By using 3.1, we conclude finally that $\alpha = \sum_{i=0}^{\infty} \frac{a_i}{(-p)^i}$ is transcendent.

4 Open Problem

If we replace $(-p)$ an integer by $(-\beta)$ for β not an integer but Pisot number. Does Theorem 3.1 remains true or not.

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