

Multiple Positive Solutions for Second Order Three Points Boundary Value problems in Cone

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Abstract

In this paper, the authors consider the following nonlinear differential equation

$$-u''(t) + k^2u(t) = f(t, u(t)), \quad t \in [0, 1]$$

with three point boundary conditions,

$$u(0) = 0, \quad u(1) = \alpha u(\eta)$$

where $k > 0, 0 < \eta < 1, \alpha \in \mathbb{R}$ $f : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$. By means of Avery-Henderson fixed point theorem and Avery-Peterson fixed point theorem, the authors establish existence of two and three positive solutions. Examples are also given to illustrate the main results.

Keywords: *Three point Boundary Value Problem, Fixed Point Theorem, Positive Solution, Green's Function, Cone.*

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1 Introduction

Multi-point boundary value problems (MPBVP's) for ordinary differential equations arise in different area of applied mathematics problems[16]. Boundary value problems(BVPs) with multi-point boundary conditions appear in

the mathematical modelling of visco-elastic and inelastic flows deformation of beams and plate deflection theory [5]. The study of MPBVP initiated by the work of Il'in and Moiseev [7, 8]. Then after by applying different fixed point theorems several authors studied the existence and multiplicity of positive solutions for nonlinear second order three point BVPs, for example, in 1992, Gupta [4] obtained the existence and uniqueness theorem for three-point BVPs.

Further, Thompson and Tisdell in 2001[21], Lian and Ge in 2006[11], Nieto in 2013[18], Nadezhda in 2016[17], studied on existence of solutions for second order differential equation with three-point boundary conditions. While, Wei, et al. in 2019[22], by using fixed point theorem studied existence of positive solution for three point BVPs. For method and applications of fixed point theorem on three point boundary value problems we refer [1, 6, 9, 10, 13, 14, 15, 19].

Here we consider the following nonlinear ordinary differential equation of second order

$$-u''(t) + k^2 u(t) = f(t, u(t)), \quad 0 \leq t \leq 1, \quad u \in [0, \infty), \quad (1)$$

subject to three point boundary conditions

$$u(0) = 0, \quad u(1) = \alpha u(\eta) \quad (2)$$

where $k > 0, 0 < \eta < 1, \alpha \in \mathbb{R}, f : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Most of the research on existence of positive solutions for three point boundary value problems (TPBVP) on the literature focused for $k = 0$. Besides that Sveikate, N. in 2016 [17], established the existence of solution for three-point BVPs of the form

$$u''(t) + k^2 u(t) + \lambda q(t) f(t, u(t)) = 0, \quad t \in (0, 1),$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta)$$

where $k > 0, 0 < \eta < 1, \alpha \in \mathbb{R}^+$, by using Quasilinearization process. Liu, et al. in 2014 [12], studied the existence, multiplicity, and nonexistence of positive solutions for the following boundary value problem

$$x''(t) + k^2 x(t) = f(t, x(t)), \quad 0 \leq t \leq 1,$$

$$u(0) = 0, \quad u(1) = \delta u(\eta)$$

by fixed point index theory. Inspired by the above work we study the TP-BVP (1)–(2). By applying Avery-Henderson and Avery-Peterson fixed point theorem the existence of two and three positive solutions can be established.

We assume the following through out this paper

$$H1. \quad f \in C([0, 1] \times [0, \infty), [0, \infty)),$$

H2. $\alpha < \frac{\sinh k}{\sinh k\eta}, k \in (0, \infty)$

The remaining part of this paper arranged as: In section 2, Green's functions, bounds of Green's functions and some preliminaries are presented. In section 3, by putting certain conditions on f existence of two positive solutions and three positive solutions established by using Avery-Henderson fixed point theorem and Avery-Peterson fixed point theorems. Finally, in section 4, we put open problem for future work.

2 Preliminaries

In this section, we construct Green's function for the corresponding homogeneous boundary value problem (BVP) (1)-(2).

Lemma 2.1. *The BVP*

$$-u''(t) + k^2 u(t) = f(t, u(t)) \quad (3)$$

$$u(0) = 0, u(1) = 0 \quad (4)$$

has a unique positive solutions

$$\begin{aligned} w(t) &= \int_0^1 H(t, s) f(s, u(s)) ds \\ w(0) &= 0, w(1) = 0, w(\eta) = \int_0^1 H(\eta, s) f(s, u(s)) ds \end{aligned} \quad (5)$$

where

$$H(t, s) = \begin{cases} \frac{\sinh kt \sinh k(1-s)}{k \sinh k}, & 0 \leq t \leq s \leq 1 \\ \frac{\sinh ks \sinh k(1-t)}{k \sinh k}, & 0 \leq s \leq t \leq 1 \end{cases} \quad (6)$$

Proof. We proof the Lemma by the help of properties of Green's function. Let $u_1 = \cosh kt$ and $u_2 = \sinh kt$ are the two linearly independent solutions of

$$-u''(t) + k^2 u(t) = 0. \quad (7)$$

Green's function for homogeneous boundary value problem (7) can be defined as

$$H(t, s) = \begin{cases} \cosh(kt)\lambda_1(s) + \sinh(kt)\lambda_2(s), & 0 \leq t \leq s \leq 1, \\ \cosh(kt)\mu_1(s) + \sinh(kt)\mu_2(s), & 0 \leq s \leq t \leq 1. \end{cases} \quad (8)$$

where the functions, $\lambda_1(s)$, $\lambda_2(s)$, $\mu_1(s)$, and $\mu_2(s)$ to be determined. From the continuity and jump discontinuity of the Green function at $s = t$, we obtain

$\mu_1(s) = \lambda_1(s) + \frac{\sinh ks}{k}$ and $\mu_2(s) = \lambda_2(s) - \frac{\cosh ks}{k}$. Putting these values in place of $\mu_1(s)$ and $\mu_2(s)$ in 8 we get

$$H(t, s) = \begin{cases} \cosh(kt)\lambda_1(s) + \sinh(kt)\lambda_2(s), & 0 \leq t \leq s \leq 1, \\ \cosh(kt)\lambda_1(s) + \sinh(kt)\lambda_2(s) + \cosh(kt)\frac{\sinh(ks)}{k} \\ - \sinh(kt)\frac{\cosh(ks)}{k}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Finally, $H(t, s)$ satisfies the boundary conditions with respect to t . Hence, λ_1 and λ_2 uniquely expressed as,

$$\lambda_1(s) = 0$$

and

$$\lambda_2(s) = \frac{\sinh k(1-s)}{k \sinh k}$$

Consequently, substituting the value of $\lambda_1(s)$, $\lambda_2(s)$, $\mu_1(s)$, and $\mu_2(s)$ on (8) we can get the Green function $H(t, s)$ for the boundary value problem as described in the Lemma. □

Lemma 2.2. *$H(t, s)$ has the following properties:*

(i) $H(t, s) \leq H(s, s)$, for all $t, s \in [0, 1]$;

(ii) $H(t, s) \geq NH(s, s)$, for all $t \in [\delta, 1 - \delta]$, $s \in [0, 1]$, $N = \frac{\sinh k\delta}{\sinh k}$, $\delta \in (0, 1/2)$.

Proof. We refer for the proof [20]. □

Lemma 2.3. *Assume H2 holds. Then the Green function for the corresponding homogeneous second order TPBVPs (1)-(2) is given by*

$$G(t, s) = H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k(\eta)} H(\eta, s) \quad (9)$$

Proof. The three point boundary value problem (1),(2) can be obtained replacing $u(1) = 0$ by $u(1) = \alpha u(\eta)$ in (4), thus we suppose the solution of the three-point boundary value problem (1), (2) can be

$$u(t) = w(t) + B_1 \sinh kt + B_2 \sinh k(1-t) \quad (10)$$

where B_1 and B_2 are constants that will be determined. From (5), (10), we know that

$$\begin{cases} u(0) = w(0) + B_1 \sinh k(0) + B_2 \sinh k(1-0) \\ u(1) = w(1) + B_1 \sinh k(1) + B_2 \sinh k(1-1) \\ u(\eta) = w(\eta) + B_1 \sinh k(\eta) + B_2 \sinh k(1-\eta) \end{cases}$$

$$\Rightarrow \begin{cases} 0 = 0 + B_1 \cdot 0 + B_2 \sinh k(1) \\ u(1) = 0 + B_1 \sinh k(1) + B_2 \sinh k(0) \\ u(\eta) = w(\eta) + B_1 \sinh k(\eta) + B_2 \sinh k(1 - \eta) \end{cases}$$

$$\Rightarrow B_2 = 0$$

$u(1) = \alpha u(\eta)$ we have $B_1 \sinh k = \alpha(w(\eta) + B_1 \sinh k(\eta))$

$$B_1 = \frac{\alpha w(\eta)}{\sinh k - \alpha \sinh k(\eta)}. \quad (11)$$

Consequently, $u(t)$ is uniquely expressed as

$$\begin{aligned} u(t) &= w(t) + B_1 \sinh kt + B_2 \sinh k(1 - t) \\ u(t) &= \int_0^1 H(t, s) f(s, u(s)) ds + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k(\eta)} \int_0^1 H(\eta, s) f(s, u(s)) ds \\ u(t) &= \int_0^1 [H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k(\eta)} H(\eta, s)] f(s, u(s)) ds \end{aligned} \quad (12)$$

Hence, the Lemma follows. $\square$

Lemma 2.4. *Green's function $G(t, s)$ satisfies the following inequalities;*

- i. $G(t, s) \geq 0, \forall t, s \in [0, 1]$;
- ii. $G(t, s) \leq DH(s, s), \forall t, s \in [0, 1]$,
- iii. $G(t, s) \geq \mathcal{M}H(s, s), \forall t \in [\delta, 1 - \delta], s \in [0, 1]$

where

$$\begin{aligned} D &= 1 + \frac{\alpha \cdot \sinh k}{\sinh k - \alpha \sinh k\eta} \\ \mathcal{M} &= \frac{\sinh k\delta}{\sinh k} \left[1 + \frac{\alpha \cdot \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right]. \end{aligned} \quad (13)$$

Proof. (i) It is obvious that $G(t, s)$ is a nonnegative since $H(t, s) \geq 0$ and $\sinh k - \alpha \sinh k\eta > 0$

(ii) consider the following cases

Case (i) if $t \leq s, \eta \leq s$

$$\begin{aligned} G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\ &\leq H(s, s) + \frac{\alpha \sinh ks}{\sinh k - \alpha \sinh k\eta} H(s, s) \\ &\leq H(s, s) \left[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta} \right] \\ &\leq D_1 H(s, s) \end{aligned} \quad (14)$$

Case(ii) If $t \leq s \leq \eta$

$$\begin{aligned}
 G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\
 &\leq H(s, s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s, s) \\
 &\leq H(s, s) \left[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\leq D_2 H(s, s).
 \end{aligned} \tag{15}$$

Case(iii) if $s \leq t, s \leq \eta$

$$\begin{aligned}
 G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\
 &\leq H(s, s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s, s) \\
 &\leq H(s, s) \left[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\leq D_3 H(s, s)
 \end{aligned} \tag{16}$$

Case(iv) if $\eta \leq s \leq t \leq 1$

$$\begin{aligned}
 G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\
 &\leq H(s, s) + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta} H(s, s) \\
 &\leq H(s, s) \left[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\leq D_4 H(s, s)
 \end{aligned} \tag{17}$$

Therefore, $G(t, s) \leq DH(s, s)$, where $D = D_1 = D_2 = D_3 = D_4$

(iii) To prove (iii) we consider the following cases:

Case(i) If $s \leq t, s \leq \eta$

$$\begin{aligned}
 G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
 &\geq NH(s, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
 &\geq NH(s, s) \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\geq H(s, s) \frac{\sinh k\delta}{\sinh k} \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\geq M_1 H(s, s).
 \end{aligned} \tag{18}$$

Case(ii) If $t \leq s, \eta \leq s$

$$\begin{aligned}
 G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
 &\geq NH(s, s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
 &\geq NH(s, s) \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\geq H(s, s) \frac{\sinh k\delta}{\sinh k} \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\geq M_2 H(s, s).
 \end{aligned} \tag{19}$$

Case(iii) If $t \leq s \leq \eta$

$$\begin{aligned}
 G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
 &\geq NH(s, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} NH(s, s) \\
 &\geq NH(s, s) \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\geq H(s, s) \frac{\sinh k\delta}{\sinh k} \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
 &\geq M_3 H(s, s).
 \end{aligned} \tag{20}$$

Case(iv) If $\eta \leq s \leq t \leq 1$

$$\begin{aligned}
G(t, s) &= H(t, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
&\geq NH(s, s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} NH(s, s) \\
&\geq NH(s, s) \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
&\geq H(s, s) \frac{\sinh k\delta}{\sinh k} \left[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta} \right] \\
&\geq M_4 H(s, s).
\end{aligned} \tag{21}$$

Therefore, $G(t, s) \geq \mathcal{M}H(s, s)$, where $\mathcal{M} = M_1 = M_2 = M_3 = M_4$

□

3 Main results

In this section we provide our main result in two subsections.

3.1 Two Positive Solutions

In this subsection, we establish the existence of at least two positive solutions for the boundary value problem (1),(2) by applying Avery-Henderson fixed point theorem in cones.

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space E . For a positive real number c , we define the set $P(\psi, c) = \{u \in P : \psi(u) < c\}$. In obtaining two positive solutions of the TPBVPs (1), (2) the following Avery-Henderson functional fixed point theorem will be the fundamental tool.

Theorem 3.1. [2] *Let P be cone in the real Banach space E . Suppose γ and ψ are increasing, nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c and M ,*

$$\gamma(u) \leq \theta(u) \leq \psi(u) \text{ and } ||u|| \leq M\gamma(u)$$

for all $u \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ such that $\theta(\lambda u) \leq \lambda\theta(u)$, for $0 \leq \lambda \leq 1$ and $u \in \partial P(\theta, b)$. Further, $T : \overline{P(\gamma, c)} \rightarrow P$ be a completely continuous operator such that

(B1) $\gamma(Tu) > c$, for all $u \in \partial P(\gamma, c)$,

(B1) $\theta(Tu) < b$, for all $u \in \partial P(\theta, b)$,

(B1) $\psi(Tu) > a$ and $P(\psi, a) \neq \emptyset$ for all $u \in \partial P(\psi, a)$.

Then T has at least two fixed points $u_1, u_2 \in \overline{P(\gamma, c)}$, such that $a < \psi(u_1)$ with $\psi(u_1) < b$ and $b < \theta(u_2)$ with $\gamma(u_2) < c$.

Let the Banach space $E = \{u : u \in C[0, 1]\}$ be equipped with norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$, and let

$$P = \{u \in E; u(t) \geq 0 \text{ and } \min_{\delta \leq t \leq 1-\delta} u(t) \geq \omega \|u\|\} \quad (22)$$

where $\omega = \frac{M}{D}$, then P is a nonempty closed subset of E . Define an operator $T : P \rightarrow E$ as

$$(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds \quad (23)$$

where $G(t, s)$ is as defined in (9). For the notational convenience, we denote $R, S, \mathcal{L}, L$ by

$$R = \frac{1}{\int_0^1 H(s, s) ds} > 0, \quad S = \frac{1}{\int_0^1 \omega H(s, s) ds} > 0 \quad (24)$$

$$\mathcal{L} = M \int_{\delta}^{1-\delta} H(s, s) ds \text{ and } L = D \int_0^1 H(s, s) ds. \quad (25)$$

By applying fixed point theorem on T and setting suitable conditions on f we determine the existence of at least two fixed points in a cone P .

Lemma 3.2. *Let H_1 and H_2 hold then the operator $T : P \rightarrow P$ is completely continuous.*

Proof. First we prove the following

1. The operator T is self map on P . Now (1)-(2) has a solution $u = u(t)$ if and only if u is the fixed point of the operator T defined in equation 23.

Now $G(t, s)$ is the Green's function for the boundary value problem, by Lemma 2.4 we have $G(t, s) \leq DH(s, s)$, for all $t, s \in [0, 1]$ and $Tu \in E$, for each $u \in P$. We have

$$Tu(t) = \int_0^1 G(t, s) f(s, u(s)) ds \leq \int_0^1 DH(s, s) f(s, u(s)) ds \quad (26)$$

and we have $Tu(t) \leq \int_0^1 DH(s, s) f(s, u(s)) ds$ which implies that

$$\|Tu\| \leq \int_0^1 DH(s, s) f(s, u(s)) ds$$

then,

$$\begin{aligned}
\min_{t \in [\delta, 1-\delta]} (Tu)(t) &= \min_{t \in [\delta, 1-\delta]} \int_0^1 G(t, s) f(s, u(s)) ds \\
&\geq \mathcal{M} \int_0^1 H(s, s) f(s, u(s)) ds = \frac{\mathcal{M}}{D} \int_0^1 DH(s, s) f(s, u(s)) ds \\
&\geq \frac{\mathcal{M}}{D} \|T(u)\| = \omega \|T(u)\|.
\end{aligned}$$

$TP \subset P$. Therefore, T is a self map on P .

2. The operator T is uniformly bounded on P . Let $u \in P$, in view of the positivity and continuity of $G(t, s)$, $H(t, s)$ and f , we have $T : P \rightarrow P$ is continuous.

Let $\{u_k\}$ be a bounded sequence in P , say $\|u_k\| \leq M$ for all k since f is continuous, there exist $N > 0$ such that $|f(t, u(t))| \leq N$ for all $u \in [0, \infty)$ with $0 \leq u \leq M$ then for each $t \in [0, 1]$ and for each k ,

$$\begin{aligned}
|Tu_k(t)| &= \left| \int_0^1 G(t, s) f(s, u_k) ds \right| \\
&\leq \int_0^1 G(t, s) (N) ds \\
&\leq N \int_0^1 G(t, s) ds < \infty
\end{aligned}$$

that is for each $t \in [0, 1]$, u_k is a bounded sequence of real numbers. By choosing successive subsequences for each t , there exist a subsequence $\{u_k\}$ which converges uniformly for $t \in [0, 1]$. Hence, T is uniformly bounded.

3. The operator T is equicontinuous on P . To prove T is equicontinuous. Let $u \in P$, and $\epsilon > 0$ be given. By the continuity of $G(t, s)$, for $t \in [0, 1]$, there exists $\delta > 0$ such that $|G(t_2, s) - G(t_1, s)| < \frac{\epsilon}{N}$ whenever $|t_1 - t_2| < \delta$, for $t_1, t_2 \in [0, 1]$

$$\begin{aligned}
\|Tu(t_1) - Tu(t_2)\| &= \left| \int_0^1 (G(t_1, s) - G(t_2, s)) f(s, u(s)) ds \right| \\
&\leq \int_0^1 |G(t_2, s) - G(t_1, s)| N ds, \\
&\leq N \int_0^1 |G(t_1, s) - G(t_2, s)| ds, \\
&< \epsilon.
\end{aligned}$$

Therefore, by a standard application of the Arzela-Ascoli theorem and the result from 1, 2 and 3, T is completely continuous.

□

Define the nonnegative, increasing, continuous functionals γ , θ , and ψ on the cone P by:- $\gamma(u) = \min_{\delta \leq t \leq 1-\delta} u(t)$
 $\theta(u) = \max_{\delta \leq t \leq 1-\delta} u(t)$, $\psi(u) = \max_{0 \leq t \leq 1} u(t)$ We observe that for any $u \in P$,

$$\gamma(u) \leq \theta(u) \leq \psi(u). \quad (27)$$

And

$$\|u\| \leq \frac{\gamma(u)}{\omega} \leq \frac{\theta(u)}{\omega} \leq \frac{\psi(u)}{\omega}. \quad (28)$$

Theorem 3.3. *Let conditions (H1) and (H2) hold, there exist real numbers $0 < a < b < c$ such that $f(t, u(t))$ satisfies the following inequalities:*

$$(A1) \quad f(t, u(t)) > \frac{cS}{D}, t \in (\delta, 1 - \delta) \text{ and } u \in (c, \frac{c}{\omega}),$$

$$(A2) \quad f(t, u(t)) < \frac{bR}{D}, t \in [0, 1] \text{ and } u \in [0, \frac{b}{\omega}],$$

$$(A3) \quad f(t, u(t)) > \frac{aS}{D}, t \in (\delta, 1 - \delta) \text{ and } u \in [a, \frac{a}{\omega}].$$

Then the three point BVP (1)-(2) has at least two positive solutions u_1 and u_2 such that $a < \psi(u_1)$ with $\psi(u_1) < b$ and $b < \theta(u_2)$ with $\gamma(u_2) < c$.

Proof. Consider the operator $T : P \rightarrow E$, where T as defined in (23). We seek two fixed points of T in P . For each $u \in P$, from (27), (28)

$$\gamma(u) \leq \theta(u) \leq \psi(u)$$

and

$$\|u\| \leq \frac{\gamma(u)}{\omega}.$$

For $u \in P$, $Tu \in P$ by first part of Lemma 3.2. Also, for any $0 \leq \lambda \leq 1$, $\theta(\lambda u) = \max_{\delta \leq t \leq 1-\delta} (\lambda u)(t) = \lambda \max_{\delta \leq t \leq 1-\delta} u(t) = \lambda \theta(u)$. It is clear that $\theta(0) = 0$. We now show that the remaining conditions of Theorem 3.1, are satisfied. First, we verify that condition(B1) of Theorem 3.1 is satisfied. Since

$$u \in \partial P(\gamma, c)$$

from (28) we have that $c = \min_{\delta \leq t \leq 1-\delta} u(t) \leq \|u\| \leq \frac{c}{\omega}$. Then

$$\begin{aligned} \gamma(Tu) &= \min_{\delta \leq t \leq 1-\delta} \int_0^1 G(t, s) f(s, u(s)) ds \\ &\geq \int_0^1 \mathcal{M}H(s, s) \frac{cS}{D} ds \\ &> cS \int_0^1 \omega H(s, s) ds = c, \end{aligned} \quad (29)$$

using condition (A1) of Theorem 3.3.

Then, we shall show that condition (B2) of Theorem 3.1 is satisfied. Since $u \in \partial P(\theta, b)$ from (28) we have that $0 \leq u(t) \leq \|u\| \leq \frac{b}{\omega}$, for $t \in [0, 1]$. Thus,

$$\begin{aligned} \theta(Tu) &= \max_{\delta \leq t \leq 1-\delta} \int_0^1 G(t, s) f(s, u(s)) ds \\ &\leq \int_0^1 DH(s, s) f(s, u(s)) ds \\ &< \int_0^1 DH(s, s) \frac{bR}{D} ds = bR \int_0^1 H(s, s) ds = b, \end{aligned} \quad (30)$$

by condition (A2) of Theorem 3.3. Finally, using condition (A3) of 3.3, to settle criteria (B3) of Theorem 3.1. Since $0 \in P$ and $a > 0$, $P(\psi, a) \neq \emptyset$. Since, $u \in \partial P(\psi, a)$, $a = \max_{0 \leq t \leq 1} u(t) \leq \|u\| \leq \frac{a}{\omega}$, for $t \in [0, 1]$. Therefore,

$$\begin{aligned} \psi(Tu) &= \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s)) ds \\ &\geq \int_0^1 \mathcal{M}H(s, s) f(s, u(s)) ds \\ &> \int_0^1 H(s, s) \mathcal{M} \frac{aS}{D} ds = aS \int_0^1 \omega H(s, s) ds = a. \end{aligned} \quad (31)$$

Thus, all the conditions of Theorem 3.1 are satisfied. Therefore, the boundary value problem (1.1), (1.2) has at least two positive solutions u_1, u_2 in cone P . Further, $a < \max_{t \in [0, 1]} u_1 < b$ and $b < \max_{t \in [\delta, 1-\delta]} u_2$ with $\min_{t \in [\delta, 1-\delta]} u_2 < c$. $\square$

Example 3.3.1. Consider the boundary value problem:

$$-u''(t) + \frac{1}{9}u(t) = f(t, u(t)), \quad 0 \leq t \leq 1 \quad (32)$$

$$u(0) = 0, u(1) = \alpha u(\eta) \quad (33)$$

where

$$f(t, u(t)) = \frac{(u+1)^5}{u^2 + 101}.$$

The Green function for the corresponding homogeneous BVP of (32)-(33) is

the particular case of Equation (9) by taking $\alpha = 2$ and $\eta = 1/5$.

$$G(t, s) = \begin{cases} \frac{(\sinh \frac{1}{3}(1-s) + 2 \sinh \frac{1}{3}(s - \frac{1}{5}))(\sinh \frac{1}{3}t)}{\frac{1}{3}(\sinh \frac{1}{3} - 2 \sinh \frac{1}{15})}, & t \leq s \leq \frac{1}{5}, \\ \frac{(\sinh \frac{1}{3}(1-t) + 2 \sinh \frac{1}{3}(t - \frac{1}{5}))(\sinh \frac{1}{3}s)}{\frac{1}{3}(\sinh \frac{1}{3} - 2 \sinh \frac{1}{15})}, & s \leq t, s \leq \frac{1}{5}, \\ \frac{\sinh \frac{1}{3}t \sinh \frac{1}{3}(1-s)}{\frac{1}{3}(\sinh \frac{1}{3} - 2 \sinh \frac{1}{15})}, & t \leq s, \frac{1}{5} \leq s, \\ \frac{\sinh \frac{1}{3}s \sinh \frac{1}{3}(1-t) + 2 \sinh \frac{1}{15} \sinh \frac{1}{3}(t-s)}{\frac{1}{3}(\sinh \frac{1}{3} - 2 \sinh \frac{1}{15})}, & \frac{1}{5} \leq s \leq t \leq 1. \end{cases}$$

By algebraic calculation and setting $\delta = \frac{1}{7}$ we obtain the following results, $N = 0.140$, $D = 4.2945$, $\mathcal{M} = 0.2047$, $\omega = \frac{\mathcal{M}}{D} = 0.0476$, $R = 0.9384$, $S = 19.716$. If $a = 0.0001$, $b = 0.0361$, and $c = 3.4846$, then $f(t, u(t))$ satisfies

$$(i) \ f(t, u(t)) > 16.0326 = \frac{cS}{D}, t \in (\frac{1}{7}, \frac{6}{7}) \text{ and } u \in (3.4846, 73.2058),$$

$$(ii) \ f(t, u(t)) < 0.1658 = \frac{bR}{D}, t \in [0, 1] \text{ and } u \in [0, 0.7592],$$

$$(iii) \ f(t, u(t)) > 4.601 \times 10^{-4} = \frac{aS}{D}, t \in (\frac{1}{7}, \frac{6}{7}) \text{ and } u \in [0.0001, 0.0022].$$

Then all conditions of Theorem 3.3 satisfied. So by Theorem 3.3 the three points boundary value problem (32)-(33) has at least two positive solutions u_1 and u_2 such that

$$0.0001 < \max_{0 \leq t \leq 1} u_1(t) \text{ with } \max_{0 \leq t \leq 1} u_1(t) < 0.0361 \text{ and}$$

$$0.0361 < \max_{1/7 \leq t \leq 6/7} u_2(t) \text{ with } \min_{1/7 \leq t \leq 6/7} u_2(t) < 3.4846.$$

3.2 Three Positive Solutions

In this subsection by applying Avery-Peterson fixed point theorem (Theorem 3.4) we investigate the existence of at least three positive solutions for TPBVPs (1), (2).

To make this work self explanatory we put the definition of the following sets as:

$$\begin{aligned} \overline{P(\gamma, d)} &= \{u \in P : \gamma(u) \leq d\} \\ P(\alpha, \gamma, b, d) &= \{u \in \overline{P(\gamma, d)} : \alpha(u) \geq b\}, \\ P(\alpha, \theta, \gamma, b, c, d) &= \{u \in \overline{P(\gamma, d)} : \alpha(u) \geq b, \theta(u) \leq c\}, \\ P(\psi, \gamma, a, d) &= \{u \in \overline{P(\gamma, d)} : \psi(u) \geq a\}. \end{aligned}$$

Theorem 3.4. [3] Let P be a cone in a real Banach space E . Let γ and θ be nonnegative continuous convex functional on P . Let α be a nonnegative continuous concave functional on P , and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d ,

$$\alpha(x) \leq \psi(x) \text{ and } \|x\| \leq M\gamma(x)$$

for $x \in \overline{P(\gamma, d)}$. Suppose that $T : \overline{P(\gamma, d)} \rightarrow \overline{P(\gamma, d)}$ is completely continuous operator and there exist positive numbers a, b and c with $a < b$ such that

C1. $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\alpha, b; \theta; c; \gamma, d)$;

C2. $\alpha(Tx) > b$ for $x \in P(\alpha, b; \gamma, d)$ with $\theta(Tx) > c$;

C3. $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that $\gamma(x_i) \leq d$ for $i = 1, 2, 3$; $b \leq \alpha(x_1)$; $a \leq \psi(x_2)$ with $\alpha(x_2) \leq b$; $a \leq \psi(x_3) \leq a$.

Let $\gamma(u) = \|u\| = \max_{0 \leq t \leq 1} (u(t))$, $t \in [0, 1]$, $\theta(u) = \psi(u) = \max_{0 \leq t \leq 1} |u(t)|$, $\alpha(u) = \min_{\eta \leq t \leq 1} u(t)$, where γ and θ are non-negative convex functional, ψ is a nonnegative functional α is a nonnegative concave functional on the cone P .

Now, we give the second result of this paper as follow.

Theorem 3.5. Let conditions (H1) and (H2) hold there exist positive numbers a, b, d , with $0 < a < b < d$, such that

A1. $f(t, u(t)) \leq \frac{d}{L}, t \in [0, 1], u \in [0, d]$,

A2. $f(t, u(t)) > \frac{b}{L}, t \in [\delta, 1 - \delta], u \in [b, \frac{b}{\omega}]$,

A3. $f(t, u(t)) < \frac{a}{L}, t \in [0, 1], u \in [0, a]$.

Then boundary value problem (1)-(2) has at least three positive solution u_1, u_2 and $u_3 \in \overline{P(\gamma, d)}$ satisfying $\gamma(u_i) \leq d$ for $i=1, 2, 3$ and $\min_{\delta \leq t \leq 1-\delta} u_1(t) > b$, $\max_{0 \leq t \leq 1} u_2(t) > a$ with $\min_{\delta \leq t \leq 1-\delta} u_2(t) < b$ and $\max_{0 \leq t \leq 1} u_3(t) < a$.

Proof. We prove to satisfies all the condition of theorem which lead as the existence of at least three fixed points of T in a cone p .

$$\gamma(u) = \max_{0 \leq t \leq 1} u(t)$$

Let $u \in \overline{P(\gamma, d)}$, then $\gamma(u) \leq d$.

$$\begin{aligned} \gamma(Tu) &= \max_{0 \leq t \leq 1} (Tu)(t) = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s)) ds \\ &\leq D \int_0^1 H(s, s) \frac{d}{L} ds \leq \frac{dD}{L} \int_0^1 H(s, s) ds \\ &\leq d. \end{aligned}$$

Hence, $Tu \in \overline{P(\gamma, d)}$.

Then, $T : \overline{P(\gamma, d)} \rightarrow \overline{P(\gamma, d)}$.

Assume, $u(t) = \frac{d}{\omega}, 0 \leq t \leq 1, \alpha(u) = \min_{\delta \leq t \leq 1-\delta} u(t) = \frac{d}{\omega} > b,$
 $\theta(u) = \frac{d}{\omega} = \max_{0 \leq t \leq 1} u(t) = \psi(u), \gamma(u) < d.$

By definition of α

$$\begin{aligned} \alpha(Tu) &= \min_{\delta \leq t \leq 1-\delta} (Tu(t)) \\ &= \min_{\delta \leq t \leq 1-\delta} \int_0^1 G(t, s) f(t, u(s)) ds \\ &\geq \mathcal{M} \int_0^1 H(s, s) f(s, u(s)) ds \\ &\geq \mathcal{M} \int_{\delta}^{1-\delta} H(s, s) \frac{b}{\mathcal{L}} ds \\ &= b \frac{\mathcal{M}}{\mathcal{L}} \int_{\delta}^{1-\delta} H(s, s) ds \\ &> b. \end{aligned}$$

This show that condition (C1) of Theorem 3.4 satisfied. From the second conditions of Theorem 3.5, $f(t, u(t)) \geq \frac{b}{\mathcal{L}}, t \in [\delta, 1 - \delta], u \in [b, \frac{b}{\omega}]$. If $u \in P(\gamma, \alpha, b, d)$ and $\theta(Tu) > \frac{b}{\omega}$, then

$$\begin{aligned} \alpha(Tu) &= \min_{\delta \leq t \leq 1-\delta} (Tu(t)) = \min_{t \in [\delta, 1-\delta]} \int_0^1 G(t, s) f(s, u(s)) ds \\ &\geq \frac{\mathcal{M}}{D} \int_0^1 DH(s, s) f(t, u(s)) ds \\ &\geq \omega \max_{0 \leq t \leq 1} Tu(t) = \omega \theta(Tu) > \frac{\omega b}{\omega} \\ &> b \end{aligned}$$

since $\theta(Tu) > \frac{b}{\omega} = c$. This show that C2 of Theorem 3.4 satisfied. To verify C3 of Theorem 3.4. Obviously $\psi(0) = 0 < a$, so $0 \notin R(\gamma, \psi, a, d)$.

Suppose that $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$, then $0 < u(t) \leq a, t \in [0, 1]$.

By (A3) we get

$$\begin{aligned}
\psi(Tu) &= \max_{0 \leq t \leq 1} Tu(t) \\
&= \max_{0 \leq t \leq 1} \left[\int_0^1 G(t, s) f(s, u(s)) ds \right] \\
&\leq D \int_0^1 H(s, s) f(s, u(s)) ds \\
&\leq D \int_0^1 H(s, s) \frac{a}{L} ds \\
&\leq a.
\end{aligned}$$

Condition C3 of Theorem 3.4 also fulfilled. By Theorem 3.4, T has at least three fixed points u_1, u_2, u_3 such that $\min_{\delta \leq t \leq 1-\delta} u_1 > b$, $\max_{0 \leq t \leq 1} u_2 > a$, $\min_{\delta \leq t \leq 1-\delta} u_2 < b$, $\max_{0 \leq t \leq 1} u_3 < a$. $\square$

Next we provide example to illustrate our main result

Example 3.5.1.

$$\begin{aligned}
-u''(t) + u(t) &= f(t, u(t)), 0 \leq t \leq 1, \\
u(0) = 0, u(1) &= 3u\left(\frac{1}{7}\right),
\end{aligned} \tag{34}$$

where $k = 1, \alpha = 3, \eta = \frac{1}{7}$

$$f(t, u(t)) = \begin{cases} u^2 t, & t \in [0, 1], u \in [0, 1], \\ u^2 t + (u - 1), & t \in [\delta, 1 - \delta], u \in [1, 1100], \\ 1.21 \times 10^6 t + 1099, & t \in [0, 1], u \in [1100, \infty]. \end{cases}$$

By the help of Equation (9) Green's function for the corresponding homogeneous BVP of (34) is

$$G(t, s) = \begin{cases} \frac{(\sinh(1-s) + 3 \sinh(s - \frac{1}{7}))(\sinh t)}{(\sinh 1 - 3 \sinh(\frac{1}{7}))}, & t \leq s \leq \frac{1}{7}, \\ \frac{(\sinh(1-t) + 3 \sinh(t - \frac{1}{7}))(\sinh s)}{(\sinh 1 - 3 \sinh(\frac{1}{7}))}, & s \leq t, s \leq \frac{1}{7} \\ \frac{\sinh t \sinh(1-s)}{(\sinh 1 - 3 \sinh(\frac{1}{7}))}, & t \leq s, \frac{1}{7} \leq s, \\ \frac{\sinh s \sinh(1-t) + 3 \sinh \frac{1}{7} \sinh(t-s)}{(\sinh 1 - 3 \sinh(\frac{1}{7}))}, & \frac{1}{7} \leq s \leq t \leq 1. \end{cases}$$

From Equation (13) and (25) by direct calculation we get

$$L = 0.8969 \quad \mathcal{L} = 0.0461 \quad \mathcal{M} = 0.433, \quad D = 5.731$$

f is continuous and increasing on $[0, \infty]$. If we choose $a = 1, b = 83$ and $d = 1100$, then all the conditions of Theorem 3.5 are satisfied. Hence by Theorem 3.5, the boundary value problem (34) has at least three positive solutions.

4 Open Problem

In this paper we only considered the second order differential equation

$$-u''(t) + k^2 u(t) = f(t, u(t)), \quad 0 \leq t \leq 1$$

subject to the three point boundary conditions

$$u(0) = 0, \quad u(1) = \alpha u(\eta) \quad 0 < \eta < 1.$$

As an open problem we said, "Can the method used for the case k^2 is replaced by the continuous function $p(t)$ subject to the integral boundary conditions." Hence the problem become

$$-u''(t) + p(t)u(t) = f(t, u(t)), \quad 0 \leq t \leq 1$$

subject to integral boundary conditions

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References

- [1] R.P. Agarawal, D. O'Regan, and P.J.Y. Wong, Positive Solutions of Differential, Difference and Integral Equations, Kluwer Academic Publisher, Dordrecht, The Netherlands, 1999.
- [2] R.I. Avery and J. Henderson, *Two positive fixed points of nonlinear operators on ordered Banach space*, Comm. App. Nonlinear Anal., 8 (2001), 27-36.
- [3] R.I. Avery, A. Peterson, *Three positive fixed points of nonlinear operators on ordered Banach spaces* Comput. Math. Appl., 42 (2001), 313-322.
- [4] C.P. Gupta, *Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation*, J. Math. Anal. Appl., 168 (1992), 540-551.
- [5] H.A. Mohamed, *Multi-point special boundary-value problems and applications to fluid flow through porous media*, Proceedings of International Multi-Conference of Engineers and Computer Scientists (IMECS 2009), Hong Kong. 31(2009).
- [6] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, USA, 1988.

- [7] V.A. IL'IN and E.I MOISEEV, *Nonlocal boundary value problems of the second kind for a Sturm-Liouville operator*, Differ. Equ., 23(1987), 979-987.
- [8] V.A. IL'IN and E.I MOISEEV, *A nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in differential and difference interpretations*, Differ. Equ., 23(1987), 803-811.
- [9] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd, Groningen, The Netherlands, 1964.
- [10] R.W. Leggett and L.R Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana University Mathematics J., 28(4) (1979).
- [11] H. Lian, W. Ge, *Solvability for second-order three-point boundary value problems on a half-line*, Applied Mathematics Lett., 19 (2006), 1000-1006.
- [12] J. Liu, H. Feng and X. Feng, *Multiplicity of Positive Solutions for a Singular Second-Order Three-Point Boundary Value Problem with a Parameter*, J. App.Math., (2014)603203.
- [13] R. Ma, *Positive solutions of a nonlinear three-point boundary value problem*, Elect. J. Diff. Equat., (1999), 1-8.
- [14] R. Ma, *Multiplicity of positive solutions for second-order three-point boundary value problems*, Comput. Math. Appl., 40(2000), 193-204.
- [15] R. Ma, *Positive solutions for second-order three-point boundary value problems*, Appl. Math. Lett., 14(2001), 1-5.
- [16] G.H. Meyer, *Initial value methods for boundary value problems: Theory and Applications of Invariant Imbedding*, Academic Press, New York, 1973.
- [17] Nadezhda Sveikate, *On Three-Point Boundary Value Problem*, Mathematical Modelling and Analysis, 21(2) (2016), 270-281.
- [18] J.J. Nieto, *Existence of a solution for a three-point boundary value problem for a second-order differential equation at resonance*, Boundary Value Problems, 2013 (2013:130).
- [19] K.R. Prasad and P. Murali, *Multiple positive solutions for nonlinear third order general three point boundary value problems*, Diff. Equat. and Dynamical Sys., 16(1, 2)(2008), 63-75.

- [20] K.R. Prasad, L.T. Wesen and N. Sreedhar, *Existence of positive solutions for second-order undamped Sturm-Liouville boundary value problems*, Asian-European Journal of Mathematics, 9(3)(2016),1-9.
- [21] H.B. Thompson and C. Tisdell, *Three-Point Boundary Value Problems For Second-Order, Ordinary, Differential Equations*, Mathematical and Computer Modelling, 34 (2001), 311-318.
- [22] Y. Wei, Z. Bai and S. Sun, *On positive solutions for some second-order three-point boundary value problems with convection term*, J. Inequa. and Appl., (2019) 2019:72.