

Lower and Upper bounds of SK indices of F -join of graphs

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Abstract

A topological index of a graph is a single unique number characteristic of the graph and is mathematically invariant under graph automorphism. In 2016, SK indices of a graph are defined by Shegehalli and Kanabur. In this paper SK indices of vertex F -join and edge F -join operations for different graph operations related to join of graphs are scrutinized.

Keywords: *graph operations, topological indices.*

1 Introduction

A systematic study of topological indices is one of the most striking aspects with its applications in many branches of mathematics and various other fields of science and technology. A topological index is a mathematical formula

giving a numeric quantity for the graph usually corresponding to a molecule saving us from money and time consuming laboratory experiments. Usage of topological indices in chemistry began in 1947 when H. Wiener developed probably the most widely known topological descriptor namely the Wiener index and used it to determine physical properties of isomers of alkanes also known as paraffins [7].

For a given graph $H = (V, E)$ where V is the vertex set and E is the edge set of H . The degree $d(x)$ of x is the number of edges of H incident to x . The minimum and maximum degrees of a graph are denoted by δ_H and Δ_H respectively.

In 1972, Gutman and Trinajstić [3, 4], defined the Zagreb indices by

$$M_1(H) = \sum_{xy \in E(H)} d_H(x) + d_H(y),$$

and

$$M_2(H) = \sum_{xy \in E(H)} d_H(x)d_H(y).$$

In [12], Shegehalli and Kanabur introduced two new degree based topological indices (SK indices) as follows:

$$SK(H) = \sum_{xy \in E(H)} \frac{d_H(x) + d_H(y)}{2}.$$

$$SK_1(H) = \sum_{xy \in E(H)} \frac{d_H(x) \times d_H(y)}{2}.$$

Note that, as usual, P_n denotes a path with n vertices and $(n - 1)$ edges whereas C_n ($n \geq 3$) denotes a cycle with n vertices. Throughout this paper, we consider simple and connected graphs.

The forthcoming section contains the preliminaries which are essential to prove our results. The main section contains the SK indices of vertex F -join and edge F -join of graphs related to different operations of graphs where $F = \{S, R, Q, T\}$ are studied.

2 Preliminaries

Let $H_1 = (V(H_1), E(H_1))$ and $H_2 = (V(H_2), E(H_2))$ be two connected graphs such that the cardinalities for the vertex and edge sets of H_1 and H_2 are n_1, n_2, m_1, m_2 , respectively. In graph theory, graph operations and derived graphs play a significant role. They help us to obtain some information on a graph and relationship made for some other graphs.

Definition 2.1. In [13], G. H. Shirdel et al., defined join $H_1 \vee H_2$ of two graphs H_1 and H_2 with disjoint vertex sets $V(H_1)$ and $V(H_2)$ is the graph on the vertex set $V(H_1) \cup V(H_2)$ and the edge set $E(H_1) \cup E(H_2) \cup \{xy \mid x \in V(H_1), y \in V(H_2)\}$. The vertices of join graph of $H_1 \vee H_2$ are having the degree as follows:

$$d_{H_1 \vee H_2}(x) = \begin{cases} d_{H_1}(x) + n_2, & \text{if } x \in V(H_1). \\ d_{H_2}(x) + n_1, & \text{if } x \in V(H_2). \end{cases}$$

Some of the definitions such as subdivision $S(H)$, semi total point $R(H)$, semi total line $Q(H)$ and total $T(H)$ are found in [1, 10].

Let $F = \{S, R, Q, T\}$. Let $I(H)$ denote the set of vertices of $F(H)$ which are inserted into each edge of H , so that $V(F(H)) = V(H) \cup I(H)$. Here we define vertex F -join and edge F -join of graph operations based on the join of two connected graphs H_1 and H_2 , as follows:

Vertex F -join of Graphs: [11] Let H_1 and H_2 be two simple graph the vertex F -join graph of H_1 and H_2 is denoted by $H_1 \dot{\vee}_F H_2$ and the graph is obtained from $F(H_1)$ and H_2 by joining each vertex of H_1 to all vertex of H_2 . The vertex and edge set of $H_1 \dot{\vee}_F H_2$ is $V(F(H_1)) \cup V(H_2)$ and $E(H_1) \cup E(H_2) \cup [xy : x \in V(H_1), y \in V(H_2)]$.

Edge F -join of Graphs: [11] Let H_1 and H_2 be two simple graph the edge F -join graph of H_1 and H_2 is denoted by $H_1 \underline{\vee}_F H_2$ and the graph is obtained from $F(H_1)$ and H_2 by joining each vertex of $I(H_1)$ with all vertex of H_2 . The vertex and edge set of $H_1 \underline{\vee}_F H_2$ is $V(F(H_1)) \cup V(H_2)$ and $E(H_1) \cup E(H_2) \cup [xy : x \in I(H_1), y \in V(H_2)]$.

Inspired by the work of Prosanta Sarkar et al. [11], in this work we will study the SK -indices of vertex F -join and edge F -join of graphs related to different operations of graphs where $F = \{S, R, Q, T\}$.

3 Main Results

In the following sub section first we consider SK and SK_1 -indices of vertex F -join of graphs respectively for different values of F as S, R, Q, T respectively. First we start with vertex S -join of graph.

3.1 Vertex F -join of graphs

In this sub section we derived closed formula of SK -indices for vertex join of graphs and using these formula we obtained SK -indices of some special graphs. The figure of vertex S -join and vertex R -join of P_3 and C_3 are given in Fig 1.

Vertex S join of graph:[11] Let H_1 and H_2 be two simple graph the vertex S -join graph of H_1 and H_2 is denoted by $H_1 \dot{\vee}_S H_2$ and the graph is obtained from $S(H_1)$ and H_2 by joining each vertex of $V(H_1)$ to all vertex of H_2 .

The vertices of vertex S -join graphs are having the degree as follows:

$$d_{H_1 \dot{\vee}_S H_2}(x) = \begin{cases} d_{H_1}(x) + n_2, & \text{if } x \in V(H_1) \\ d_{H_2}(x) + n_1, & \text{if } x \in V(H_2) \\ 2, & \text{if } x \in I(H_1). \end{cases} \quad (1)$$

Theorem 3.1. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK(H_1 \dot{\vee}_s H_2)$ is given by*

$$\begin{aligned} SK(H_1 \dot{\vee}_s H_2) &\geq \frac{1}{2}[n_1 n_2 (\delta_{H_1} + \delta_{H_2} + n_1 + n_2)] + m_1(\delta_{H_1} + n_2 + 2) + m_2(\delta_{H_2} + n_1). \\ SK(H_1 \dot{\vee}_s H_2) &\leq \frac{1}{2}[n_1 n_2 (\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2)] + m_1(\Delta_{H_1} + n_2 + 2) + m_2(\Delta_{H_2} + n_1). \end{aligned}$$

Proof. By using equation(1) in definition of SK index, we have

$$\begin{aligned} SK(H_1 \dot{\vee}_s H_2) &= \sum_{xy \in E(H_1 \dot{\vee}_s H_2)} \frac{d_{(H_1 \dot{\vee}_s H_2)}(x) + d_{(H_1 \dot{\vee}_s H_2)}(y)}{2} \\ &= 2m_1 \left(\frac{2 + d_{H_1}(y) + n_2}{2} \right) + n_1 n_2 \left(\frac{d_{H_1}(x) + n_2 + d_{H_2}(y) + n_1}{2} \right) \\ &\quad + m_2 \left(\frac{d_{H_2}(x) + n_1 + d_{H_2}(y) + n_1}{2} \right). \end{aligned}$$

Since, for a graph H , for all $x \in V(H)$, $d_H(x) \geq \delta_H$ and $d_H(x) \leq \Delta_H$. Therefore, using these facts, we have

$$\begin{aligned} SK(H_1 \dot{\vee}_s H_2) &\geq \frac{1}{2}[n_1 n_2 (\delta_{H_1} + \delta_{H_2} + n_1 + n_2)] + m_1(\delta_{H_1} + n_2 + 2) + m_2(\delta_{H_2} + n_1). \\ SK(H_1 \dot{\vee}_s H_2) &\leq \frac{1}{2}[n_1 n_2 (\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2)] + m_1(\Delta_{H_1} + n_2 + 2) + m_2(\Delta_{H_2} + n_1). \end{aligned}$$

□

Note: Throughout this paper, equality holds only if both H_1 and H_2 are regular graphs.

Figure 1: The example of $P_3 \dot{\vee}_S C_3$ and $P_3 \dot{\vee}_R C_3$ graphs

Example 3.2. using theorem 3.1, we get

$$\begin{aligned}
 (i) SK(P_{n_1} \dot{\vee}_s P_{n_2}) &\geq \frac{1}{2} [n_1 n_2 (2 + n_1 + n_2)] + m_1 (3 + n_2) + m_2 (1 + n_1). \\
 SK(P_{n_1} \dot{\vee}_s P_{n_2}) &\leq \frac{1}{2} [n_1 n_2 (4 + n_1 + n_2)] + m_1 (4 + n_2) + m_2 (2 + n_1). \\
 (ii) SK(P_{n_1} \dot{\vee}_s C_{n_2}) &> \frac{1}{2} [n_1 n_2 (3 + n_1 + n_2)] + m_1 (3 + n_2) + m_2 (2 + n_1). \\
 SK(P_{n_1} \dot{\vee}_s C_{n_2}) &< \frac{1}{2} [n_1 n_2 (4 + n_1 + n_2)] + m_1 (4 + n_2) + m_2 (2 + n_1). \\
 (iii) SK(C_{n_1} \dot{\vee}_s P_{n_2}) &> \frac{1}{2} [n_1 n_2 (4 + n_1 + n_2)] + m_1 (4 + n_2) + m_2 (2 + n_1). \\
 SK(C_{n_1} \dot{\vee}_s P_{n_2}) &< \frac{1}{2} [n_1 n_2 (3 + n_1 + n_2)] + m_1 (4 + n_2) + m_2 (1 + n_1). \\
 (iv) SK(C_{n_1} \dot{\vee}_s C_{n_2}) &= \frac{1}{2} [n_1 n_2 (4 + n_1 + n_2)] + m_1 (4 + n_2) + m_2 (2 + n_1).
 \end{aligned}$$

Theorem 3.3. Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK_1(H_1 \dot{\vee}_s H_2)$ is given by

$$\begin{aligned}
 SK_1(H_1 \dot{\vee}_s H_2) &\geq \frac{1}{2} \{4m_1(\delta_{H_1} + n_2) + (\delta_{H_2} + n_1)[n_1 n_2(\delta_{H_1} + n_2 + m_2)(\delta_{H_2} + n_1)]\}. \\
 SK_1(H_1 \dot{\vee}_s H_2) &\leq \frac{1}{2} \{4m_1(\Delta_{H_1} + n_2) + (\Delta_{H_2} + n_1)[n_1 n_2(\Delta_{H_1} + n_2 + m_2)(\Delta_{H_2} + n_1)]\}.
 \end{aligned}$$

Proof. By using equation(1) in definition of SK_1 index, we obtain the required result $\square$

Vertex R join of graph:[11] Let H_1 and H_2 be two simple graph the vertex R -join graph of H_1 and H_2 is denoted by $H_1 \dot{\vee}_R H_2$ and the graph is obtained from $R(H_1)$ and H_2 by joining each vertex of $V(H_1)$ to all vertex of H_2 .

The vertices of R -join graphs are having the degree as follows:

$$d_{H_1 \dot{\vee}_R H_2}(x) = \begin{cases} 2d_{H_1}(x) + n_2, & \text{if } x \in V(H_1) \\ d_{H_2}(x) + n_1, & \text{if } x \in V(H_2) \\ 2, & \text{if } x \in I(H_1). \end{cases} \quad (2)$$

Theorem 3.4. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK(H_1 \dot{\vee}_R H_2)$ is given by*

$$\begin{aligned} SK(H_1 \dot{\vee}_R H_2) &\geq \frac{1}{2}[n_1 n_2 (2\delta_{H_1} + \delta_{H_2} + n_1 + n_2) + 2m_2(\delta_{H_2} + n_1) + 4m_1(2\delta_{H_1} + n_2 + 1)]. \\ SK(H_1 \dot{\vee}_R H_2) &\leq \frac{1}{2}[n_1 n_2 (2\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2) + 2m_2(\Delta_{H_2} + n_1) + 4m_1(2\Delta_{H_1} + n_2 + 1)]. \end{aligned}$$

Proof. By using equation(2) in definition of SK index, we have

$$\begin{aligned} SK(H_1 \dot{\vee}_R H_2) &= \sum_{xy \in E(H_1 \dot{\vee}_R H_2)} \frac{d_{(H_1 \dot{\vee}_R H_2)}(x) + d_{(H_1 \dot{\vee}_R H_2)}(y)}{2} \\ &= 2m_1 \left(\frac{2 + d_{H_1}(y) + n_2}{2} \right) + m_1 \left(\frac{2d_{H_1}(x) + n_2 + 2d_{H_1}(y) + n_2}{2} \right) \\ &\quad + m_2 \left(\frac{d_{H_2}(x) + n_1 + d_{H_2}(y) + n_1}{2} \right) + n_1 n_2 \left(\frac{2d_{H_1}(x) + n_2 + d_{H_2}(y) + n_1}{2} \right). \end{aligned}$$

Since, for a graph H , for all $x \in V(H)$, $d_H(x) \geq \delta_H$ and $d_H(x) \leq \Delta_H$.

Therefore, using these facts, we have

$$\begin{aligned} SK(H_1 \dot{\vee}_R H_2) &\geq \frac{1}{2}[n_1 n_2 (2\delta_{H_1} + \delta_{H_2} + n_1 + n_2) + 2m_2(\delta_{H_2} + n_1) + 4m_1(2\delta_{H_1} + n_2 + 1)]. \\ SK(H_1 \dot{\vee}_R H_2) &\leq \frac{1}{2}[n_1 n_2 (2\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2) + 2m_2(\Delta_{H_2} + n_1) + 4m_1(2\Delta_{H_1} + n_2 + 1)]. \end{aligned}$$

□

Example 3.5. *using Theorem 3.4, we get*

$$\begin{aligned} (i) SK(P_{n_1} \dot{\vee}_R P_{n_2}) &\geq \frac{1}{2}\{n_1 n_2 (n_1 + n_2 + 3) + 2(n_2 - 1)(n_1 + 1) + 4(n_1 - 1)(n_2 + 3)\}. \\ SK(P_{n_1} \dot{\vee}_R P_{n_2}) &\leq \frac{1}{2}\{n_1 n_2 (n_1 + n_2 + 6) + 2(n_2 - 1)(n_1 + 2) + 4(n_1 - 1)(n_2 + 5)\}. \\ (ii) SK(P_{n_1} \dot{\vee}_R C_{n_2}) &> \frac{1}{2}\{n_1 n_2 (n_1 + n_2 + 4) + 2n_2(n_1 + 2) + 4(n_1 - 1)(n_2 + 3)\}. \\ SK(P_{n_1} \dot{\vee}_R C_{n_2}) &< \frac{1}{2}\{n_1 n_2 (n_1 + n_2 + 6) + 2n_2(n_1 + 2) + 4(n_1 - 1)(n_2 + 5)\}. \\ (iii) SK(C_{n_1} \dot{\vee}_R P_{n_2}) &> \frac{1}{2}\{n_1 n_2 (n_1 + n_2 + 5) + 2(n_2 - 1)(n_1 + 1) + 4n_1(n_2 + 5)\}. \\ SK(C_{n_1} \dot{\vee}_R P_{n_2}) &< \frac{1}{2}\{n_1 n_2 (n_1 + n_2 + 6) + 2(n_2 - 1)(n_1 + 2) + 4n_1(n_2 + 5)\}. \\ (iv) SK(C_{n_1} \dot{\vee}_R C_{n_2}) &= \frac{1}{2}\{n_1 n_2 (n_1 + n_2 + 6) + 2(n_2 + 2) + 4n_1(n_2 + 5)\}. \end{aligned}$$

Theorem 3.6. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK_1(H_1 \dot{\vee}_R H_2)$ is given by*

$$SK_1(H_1 \dot{\vee}_R H_2) \geq \frac{1}{2}[2\delta_{H_1} + n_2[m_1(2\delta_{H_1} + n_2 + 4) + n_1 n_2(\delta_{H_2} + n_1)] + m_2(\delta_{H_2} + n_1)^2].$$

$$SK_1(H_1 \dot{\vee}_R H_2) \leq \frac{1}{2}[2\Delta_{H_1} + n_2[m_1(2\Delta_{H_1} + n_2 + 4) + n_1 n_2(\Delta_{H_2} + n_1)] + m_2(\Delta_{H_2} + n_1)^2].$$

Proof. By using equation(2) in definition of SK_1 index, we obtain the required result. $\square$

Vertex Q join of graph:[11] Let H_1 and H_2 be two simple graph the vertex Q -join graph of H_1 and H_2 is denoted by $H_1 \dot{\vee}_Q H_2$ and the graph is obtained from $Q(H_1)$ and H_2 by joining each vertex of $V(H_1)$ to all vertex of H_2 .

The vertices of Q -join graphs are having the degree as follows:

$$d_{H_1 \dot{\vee}_Q H_2}(x) = \begin{cases} d_{H_1}(x) + n_2, & \text{if } x \in V(H_1) \\ d_{H_2}(x) + n_1, & \text{if } x \in V(H_2) \\ d_{H_1}(w) + d_{H_1}(k), & \text{if } e = (w, k)e \in I(H_1). \end{cases} \quad (3)$$

Theorem 3.7. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK(H_1 \dot{\vee}_Q H_2)$ is given by*

$$SK(H_1 \dot{\vee}_Q H_2) \geq \frac{n_1 n_2}{2}(\delta_{H_1} + \delta_{H_2} + n_1 + n_2) + m_1[2(\delta_{H_1})^2 + \delta_{H_1} + n_2] + m_2(\delta_{H_2} + n_1).$$

$$SK(H_1 \dot{\vee}_Q H_2) \leq \frac{n_1 n_2}{2}(\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2) + m_1[2(\Delta_{H_1})^2 + \Delta_{H_1} + n_2] + m_2(\Delta_{H_2} + n_1).$$

Proof. By using equation(3) in definition of SK index, we have

$$\begin{aligned} SK(H_1 \dot{\vee}_Q H_2) &= \sum_{xy \in E(H_1 \dot{\vee}_Q H_2)} \frac{d_{(H_1 \dot{\vee}_Q H_2)}(x) + d_{(H_1 \dot{\vee}_Q H_2)}(y)}{2} \\ &= m_2 \left(\frac{d_{H_2}(x) + n_1 + d_{H_2}(y) + n_1}{2} \right) + 2m_1 \left(\frac{d_{H_1}(w) + d_{H_1}(k) + d_{H_1}(y) + n_2}{2} \right) \\ &\quad + m_1 \left(\frac{d_{H_1}(w) + d_{H_1}(k) - 2}{2} \right) \left(\frac{d_{H_1}(w) + d_{H_1}(k) + d_{H_1}(w) + d_{H_1}(k)}{2} \right) \\ &\quad + n_1 n_2 \left(\frac{d_{H_1}(x) + n_2 + d_{H_2}(y) + n_1}{2} \right). \end{aligned}$$

Since, for a graph H , for all $x \in V(H)$, $d_H(x) \geq \delta_H$ and $d_H(x) \leq \Delta_H$.

Therefore, using these facts, we have

$$SK(H_1 \dot{\vee}_Q H_2) \geq \frac{n_1 n_2}{2}(\delta_{H_1} + \delta_{H_2} + n_1 + n_2) + m_1[2(\delta_{H_1})^2 + \delta_{H_1} + n_2] + m_2(\delta_{H_2} + n_1).$$

$$SK(H_1 \dot{\vee}_Q H_2) \leq \frac{n_1 n_2}{2}(\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2) + m_1[2(\Delta_{H_1})^2 + \Delta_{H_1} + n_2] + m_2(\Delta_{H_2} + n_1).$$

$\square$



Figure 2: The example of $P_3 \dot{\vee}_Q C_3$ and $P_3 \dot{\vee}_T C_3$ graphs

Theorem 3.8. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK_1(H_1 \dot{\vee}_Q H_2)$ is given by*

$$SK_1(H_1 \dot{\vee}_Q H_2) \geq \frac{\delta_{H_2} + n_1}{2} [m_2(\delta_{H_2} + n_1) + n_1 n_2 (\delta_{H_1} + n_2)] + 2m_1 \delta_{H_1} (2\delta_{H_1} + n_2 - 1).$$

$$SK_1(H_1 \dot{\vee}_Q H_2) \leq \frac{\Delta_{H_2} + n_1}{2} [m_2(\Delta_{H_2} + n_1) + n_1 n_2 (\Delta_{H_1} + n_2)] + 2m_1 \Delta_{H_1} (2\Delta_{H_1} + n_2 - 1).$$

Proof. By using equation(3) in definition of SK_1 index, we obtain the required result. $\square$

Vertex T join of graph:[11] Let H_1 and H_2 be two simple graph the vertex T -join graph of H_1 and H_2 is denoted by $H_1 \dot{\vee}_T H_2$ and the graph is obtained from $T(H_1)$ and H_2 by joining each vertex of $V(H_1)$ to all vertex of H_2 .

The vertices of T -join graphs are having the degree as follows:

$$d_{H_1 \dot{\vee}_T H_2}(x) = \begin{cases} 2d_{H_1}(x) + n_2, & \text{if } x \in V(H_1) \\ d_{H_2}(x) + n_1, & \text{if } x \in V(H_2) \\ d_{H_1}(w) + d_{H_1}(k), & \text{if } e = (w, k) \in I(H_1). \end{cases} \quad (4)$$

The figure of vertex Q -join and vertex T -join of P_3 and C_3 are given in Fig 2.

Theorem 3.9. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK(H_1 \dot{\vee}_T H_2)$ is given by*

$$SK(H_1 \dot{\vee}_T H_2) \geq \frac{n_1 n_2}{2} (2\delta_{H_1} + \delta_{H_2} + n_1 + n_2) + 2m_1 [\delta_{H_1}^2 + 2\delta_{H_1} + n_2] + m_2 (\delta_{H_2} + n_1).$$

$$SK(H_1 \dot{\vee}_T H_2) \leq \frac{n_1 n_2}{2} (2\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2) + 2m_1 [\Delta_{H_1}^2 + 2\Delta_{H_1} + n_2] + m_2 (\Delta_{H_2} + n_1).$$

Proof. By using equation(4) in definition of SK index, we have

$$\begin{aligned}
SK(H_1 \dot{\vee}_T H_2) &= \sum_{xy \in E(H_1 \dot{\vee}_T H_2)} \frac{d_{(H_1 \dot{\vee}_T H_2)}(x) + d_{(H_1 \dot{\vee}_T H_2)}(y)}{2} \\
&= m_2 \left(\frac{d_{H_2}(x) + n_1 + d_{H_2}(y) + n_1}{2} \right) + 2m_1 \left(\frac{d_{H_1}(w) + d_{H_1}(k) + 2d_{H_1}(y) + n_2}{2} \right) \\
&\quad + m_1 \left(\frac{d_{H_1}(w) + d_{H_1}(k) - 2}{2} \right) \left(\frac{d_{H_1}(w) + d_{H_1}(k) + d_{H_1}(w) + d_{H_1}(k)}{2} \right) \\
&\quad + n_1 n_2 \left(\frac{d_{H_1}(x) + n_2 + d_{H_2}(y) + n_1}{2} \right) + m_1 \left(\frac{2d_{H_1}(u) + n_2 + 2d_{H_1}(v) + n_2}{2} \right).
\end{aligned}$$

Since, for a graph H , for all $x \in V(H)$, $d_H(x) \geq \delta_H$ and $d_H(x) \leq \Delta_H$.
Therefore, using these facts, we have

$$\begin{aligned}
SK(H_1 \dot{\vee}_T H_2) &\geq \frac{n_1 n_2}{2} (2\delta_{H_1} + \delta_{H_2} + n_1 + n_2) + 2m_1 [\delta_{H_1}^2 + 2\delta_{H_1} + n_2] + m_2 (\delta_{H_2} + n_1). \\
SK(H_1 \dot{\vee}_T H_2) &\leq \frac{n_1 n_2}{2} (2\Delta_{H_1} + \Delta_{H_2} + n_1 + n_2) + 2m_1 [\Delta_{H_1}^2 + 2\Delta_{H_1} + n_2] + m_2 (\Delta_{H_2} + n_1).
\end{aligned}$$

□

Theorem 3.10. Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK_1(H_1 \dot{\vee}_T H_2)$ is given by

$$\begin{aligned}
SK_1(H_1 \dot{\vee}_T H_2) &\geq \frac{\delta_{H_2} + n_1}{2} \left[m_2 (\delta_{H_2} + n_1) + \frac{m_1}{2} (2\delta_{H_1} + n_2) (6\delta_{H_1} + n_2) + n_2 \right] + 2m_1 (\delta_{H_1} - 1) \delta_{H_1}^2. \\
SK_1(H_1 \dot{\vee}_T H_2) &\leq \frac{\Delta_{H_2} + n_1}{2} \left[m_2 (\Delta_{H_2} + n_1) + \frac{m_1}{2} (2\Delta_{H_1} + n_2) (6\Delta_{H_1} + n_2) + n_2 \right] \\
&\quad + 2m_1 (\Delta_{H_1} - 1) \Delta_{H_1}^2.
\end{aligned}$$

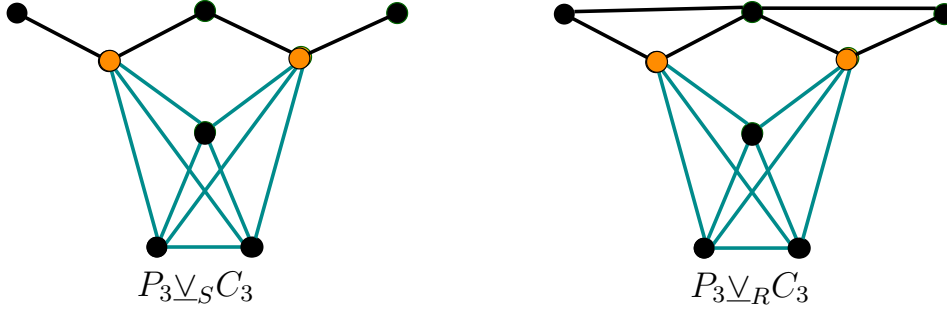
Proof. By using equation(4) in definition of SK_1 index, we obtain the required result. □

In the following subsection we consider SK -indices of edge F -join of graphs for different values of $F = \{S, R\}$.

3.2 Edge F -join of graphs

In this sub section we consider SK and SK_1 -indices of edge F -join of graphs for different values of $F = S, R$. First we start with Edge S -join of graph. The figure of edge S -join and edge R -join of P_3 and C_3 are given in Fig 3.

Edge S join of graph: Let H_1 and H_2 be two simple graph the edge S -join graph of H_1 and H_2 is denoted by $H_1 \underline{\vee}_S H_2$ and the graph is obtained from

Figure 3: The example of $P_3 \underline{\vee}_S C_3$ and $P_3 \underline{\vee}_R C_3$ graphs.

$S(H_1)$ and H_2 by joining each vertex of $I(H_1)$ to all vertex of H_2 .

The vertices of edge S -join graphs are having the degree as follows:

$$d_{H_1 \underline{\vee}_S H_2}(x) = \begin{cases} d_{H_2}(x), & \text{if } x \in V(H_1) \\ d_{H_2}(x) + m_1, & \text{if } x \in V(H_2) \\ 2 + n_2, & \text{if } x \in I(H_1). \end{cases} \quad (5)$$

Theorem 3.11. Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK(H_1 \underline{\vee}_S H_2)$ is given by

$$SK(H_1 \underline{\vee}_S H_2) \geq \frac{1}{2} [m_1(2\delta_{H_1} + 4 + n_2(\delta_{H_2} + m_1 + n_2 + 4)) + 2m_2(\delta_{H_2} + m_1)].$$

$$SK(H_1 \underline{\vee}_S H_2) \leq \frac{1}{2} [m_1(2\Delta_{H_1} + 4 + n_2(\Delta_{H_2} + m_1 + n_2 + 4)) + 2m_2(\Delta_{H_2} + m_1)].$$

Proof. By using equation(5) in definition of SK index, we have

$$\begin{aligned} SK(H_1 \underline{\vee}_S H_2) &= \sum_{xy \in E(H_1 \underline{\vee}_S H_2)} \frac{d_{(H_1 \underline{\vee}_S H_2)}(x) + d_{(H_1 \underline{\vee}_S H_2)}(y)}{2} \\ &= 2m_1 \left(\frac{d_{H_1}(x) + 2 + n_2}{2} \right) + m_1 n_2 \left(\frac{2 + n_2 + d_{H_2}(y) + m_1}{2} \right) \\ &\quad + m_2 \left(\frac{d_{H_2}(x) + m_1 + d_{H_2}(y) + m_1}{2} \right). \end{aligned}$$

Since, for a graph H , for all $x \in V(H)$, $d_H(x) \geq \delta_H$ and $d_H(x) \leq \Delta_H$. Therefore, using these facts, we have

$$SK(H_1 \underline{\vee}_S H_2) \geq \frac{1}{2} [m_1(2\delta_{H_1} + 4 + n_2(\delta_{H_2} + m_1 + n_2 + 4)) + 2m_2(\delta_{H_2} + m_1)].$$

$$SK(H_1 \underline{\vee}_S H_2) \leq \frac{1}{2} [m_1(2\Delta_{H_1} + 4 + n_2(\Delta_{H_2} + m_1 + n_2 + 4)) + 2m_2(\Delta_{H_2} + m_1)].$$

□

Theorem 3.12. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK_1(H_1 \underline{\vee}_s H_2)$ is given by*

$$\begin{aligned} SK_1(H_1 \underline{\vee}_s H_2) &\geq \frac{1}{2} [(\delta_{H_2} + m_1)[m_2 \delta_{H_2} + m_1(m_2 + 2n_2 + n_2^2)] + 2m_1 \delta_{H_1}(2 + n_2)]. \\ SK_1(H_1 \underline{\vee}_s H_2) &\leq \frac{1}{2} [(\Delta_{H_2} + m_1)[m_2 \Delta_{H_2} + m_1(m_2 + 2n_2 + n_2^2)] + 2m_1 \Delta_{H_1}(2 + n_2)]. \end{aligned}$$

Proof. By using equation(5) in definition of SK_1 index, we obtain the required result. $\square$

Edge R join of graph: Let H_1 and H_2 be two simple graph the edge R -join graph of H_1 and H_2 is denoted by $H_1 \underline{\vee}_R H_2$ and the graph is obtained from $R(H_1)$ and H_2 by joining each vertex of $I(H_1)$ to all vertex of H_2 . The vertices of edge R -join graphs are having the degree as follows:

$$d_{H_1 \underline{\vee}_R H_2}(x) = \begin{cases} 2d_{H_1}(x), & \text{if } x \in V(H_1) \\ d_{H_2}(x) + m_1, & \text{if } x \in V(H_2) \\ 2 + n_2, & \text{if } x \in I(H_1). \end{cases} \quad (6)$$

Theorem 3.13. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK(H_1 \underline{\vee}_R H_2)$ is given by*

$$\begin{aligned} SK(H_1 \underline{\vee}_R H_2) &\geq \frac{1}{2} m_1 [2(2\delta_{H_1} + n_2 + 2) + 4\delta_{H_1} + n_2(\delta_{H_2} + m_1 + n_2 + 2)] + m_2(\delta_{H_2} + m_1). \\ SK(H_1 \underline{\vee}_R H_2) &\leq \frac{1}{2} m_1 [2(2\Delta_{H_1} + n_2 + 2) + 4\Delta_{H_1} + n_2(\Delta_{H_2} + m_1 + n_2 + 2)] + m_2(\Delta_{H_2} + m_1). \end{aligned}$$

Proof. By using equation(6) in definition of SK index, we have

$$\begin{aligned} SK(H_1 \underline{\vee}_R H_2) &= \sum_{xy \in E(H_1 \underline{\vee}_R H_2)} \frac{d_{(H_1 \underline{\vee}_R H_2)}(x) + d_{(H_1 \underline{\vee}_R H_2)}(y)}{2} \\ &= 2m_1 \left(\frac{2d_{H_1}(x) + 2 + n_2}{2} \right) + m_1 n_2 \left(\frac{2 + n_2 + d_{H_2}(y) + m_1}{2} \right) \\ &\quad + m_2 \left(\frac{d_{H_2}(x) + m_1 + d_{H_2}(y) + m_1}{2} \right) + m_1 \left(\frac{2d_{H_1}(x) + 2d_{H_1}(y)}{2} \right). \end{aligned}$$

Since, for a graph G , for all $x \in V(G)$, $d_G(x) \geq \delta_G$ and $d_G(x) \leq \Delta_G$.

Therefore, using these facts, we have

$$\begin{aligned} SK(H_1 \underline{\vee}_R H_2) &\geq \frac{1}{2} m_1 [2(2\delta_{H_1} + n_2 + 2) + 4\delta_{H_1} + n_2(\delta_{H_2} + m_1 + n_2 + 2)] + m_2(\delta_{H_2} + m_1). \\ SK(H_1 \underline{\vee}_R H_2) &\leq \frac{1}{2} m_1 [2(2\Delta_{H_1} + n_2 + 2) + 4\Delta_{H_1} + n_2(\Delta_{H_2} + m_1 + n_2 + 2)] + m_2(\Delta_{H_2} + m_1). \end{aligned}$$

$\square$

Theorem 3.14. *Let H_1 and H_2 be two connected graphs. Then, the lower and upper bounds for the $SK_1(H_1 \underline{\vee}_R H_2)$ is given by*

$$SK_1(H_1 \underline{\vee}_R H_2) \geq \frac{(\delta_{H_1} + m_1)}{2} \left[m_1 n_2 (2 + n_2) + m_2 (\delta_{H_2} + m_1) \right] + 2m_1 \delta_{H_1} (2 + n_2 + \delta_{H_1}).$$

$$SK_1(H_1 \underline{\vee}_R H_2) \leq \frac{(\Delta_{H_1} + m_1)}{2} \left[m_1 n_2 (2 + n_2) + m_2 (\Delta_{H_2} + m_1) \right] + 2m_1 \Delta_{H_1} (2 + n_2 + \Delta_{H_1}).$$

Proof. By using equation 6 in definition of SK_1 index, we obtain the required result. $\square$

4 Remark

The inequalities does not satisfy for the operators $Q(H)$ and $T(H)$.

5 Conclusion

In this work, we established some useful formula of the SK -indices of graphs based on the vertex F -join and edge F -join of graphs where $F = \{S, R, Q, T\}$. Also, we derived some useful examples of the SK -indices for some important classes of graphs.

6 Open Problem

As a motivation of this work, it can be extended to find the novel topological indices for vertex F -join and edge F -join of graphs .

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