

A Note on generalized derivations

M. S. Tammam El-Sayiad

Department of Mathematics and Computer Science, Faculty of Science,
Beni-Suef University, Beni-Suef City,
P.O.Box 62111 Egypt
e-mail: mtammam2012@yahoo.com

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Abstract

In the present work we consider a prime ring S and a generalized derivation $\psi : S \rightarrow S$ associated with a derivation ϕ of S and a nonzero left ideal U of S . We prove that if S admits a generalized derivation ψ satisfying $[\psi(r), \phi(s)] = \pm\phi([s, r])$ for all $r, s \in U$, then S is commutative or $\phi(U)\phi(U) = U\phi(U) = \{0\}$.

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1 Introduction

Over the past few decades, many authors have studied the problem of finding certain specific types of algebraic conditions that force a ring to be commutative. In particular it was in 1995 [1] that Bell and Daif established a new condition that force a prime ring to be commutative, their condition could be written as $[\phi(r), \phi(s)] = \pm\phi([s, r])$ for all $r, s \in U$, where ϕ is a nonzero derivation on the ring and U a nonzero right ideal of the ring. Generalized derivations have been primarily defined by Brešar [3]. In [5], Hvala initiated the algebraic study of generalized derivations and extended some results concerning derivation to generalized derivation. Motivated by the above condition of Bell and Daif, in this paper our aim is to give a partial generalization to this condition in terms of a generalized derivation.

2 Preliminaries

Let S denote an associative ring with center Z , and let T be a nonempty subset of S . A mapping $f : S \rightarrow S$ is called centralizing on T , if $[f(r), r] \in Z$ for all $r \in T$. For any $r, s \in S$ the symbol $[r, s]$ stands for the commutator $rs - sr$. Recall that a ring S is called prime if for any $a, b \in S$, $aSb = (0)$ implies that either $a = 0$ or $b = 0$. An additive mapping $\phi : S \rightarrow S$ is called a derivation $\phi(rs) = \phi(r)s + r\phi(s)$ holds for all $r, s \in S$. An additive mapping $\psi : S \rightarrow S$ is called a generalized derivation if there exists a derivation $\phi : S \rightarrow S$ such that $\psi(rs) = \psi(r)s + r\phi(s)$ holds for all $r, s \in S$. Throughout the present paper, we shall make use of the two basic commutator identities $[r, st] = s[r, t] + [r, s]t$ and $[rs, t] = [r, t]s + r[s, t]$. Moreover we shall require the following two results.

Lemma 2.1. [4] *Let S be a prime ring and suppose that $a \in S$ centralizes a nonzero left ideal U of S . Then $a \in Z$.*

Lemma 2.2. [2] *Let S be a prime ring and U a nonzero left ideal. If S admits a nonzero derivation which is centralizing on U , then S is commutative.*

3 Main results

Theorem 3.1. *Let S be a prime ring and U a nonzero left ideal. If S admits a nonzero generalized derivation ψ associated with a nonzero derivation ϕ satisfying $[\psi(r), \phi(s)] = \phi([s, r])$ for all $r, s \in U$, then S is commutative or $\phi(U)\phi(U) = U\phi(U) = \{0\}$.*

Proof. We have

$$[\psi(r), \phi(s)] = \phi([s, r]), \quad \text{for all } r, s \in U. \quad (1)$$

Substituting sr for s in (1), we get

$$\phi(s)[\psi(r), r] + [\psi(r), s]\phi(r) = [s, r]\phi(r), \quad \text{for all } r, s \in U. \quad (2)$$

Replacing s by rs in (2) and using (2), we find that

$$[\psi(r), r]s\phi(r) + \phi(r)s[\psi(r), r] = 0, \quad \text{for all } r, s \in U. \quad (3)$$

In (2) we substitute $\psi(r)s$ for s , since U is a left ideal, to get

$$[\psi(r), r]s\phi(r) - \phi(\psi(r))s[\psi(r), r] = 0, \quad \text{for all } r, s \in U. \quad (4)$$

From (3) and (4) we obtain

$$\{\phi(r) + \phi(\psi(r))\}s[\psi(r), r] = 0, \quad \text{for all } r, s \in U. \quad (5)$$

Thus, (5) yields

$$\{\phi(r) + \phi(\psi(r))\}SU[\psi(r), r] = \{0\}, \quad \text{for all } r \in U. \quad (6)$$

But S is prime, hence for each $r \in U$, we have either $U[\psi(r), r] = \{0\}$ or $\phi(r) + \phi(\psi(r)) = 0$. The sets of r for which these conditions hold are additive subgroups of U with union equal to U ; hence either $U[\psi(r), r] = \{0\}$ for all $r \in U$ or $\phi(r) + \phi(\psi(r)) = 0$ for all $r \in U$.

If $U[\psi(r), r] = \{0\}$ then (4) shows that

$$[\psi(r), r]s\phi(r) = 0, \quad \text{for all } r, s \in U.$$

So that

$$[\psi(r), r]SU\phi(r) = \{0\}, \quad \text{for all } r \in U.$$

Therefore, for each $r \in U$ either $[\psi(r), r] = 0$ or $U\phi(r) = \{0\}$. But again the sets of r for which these conditions hold are additive subgroups of U with union equal to U . So either $U\phi(U) = \{0\}$ or $[\psi(r), r] = 0$ for all $r \in U$. In the latter case we have

$$[\psi(r), r] = 0, \quad \text{for all } r \in U. \quad (7)$$

Polarizing this condition by replacing r by $r + s$, we obtain

$$[\psi(r), s] + [\psi(s), r] = 0, \quad \text{for all } r, s \in U. \quad (8)$$

Now by substituting $s = sr$ and using (7) and (8) we arrive at

$$[s\phi(r), r] = 0, \quad \text{for all } r, s \in U.$$

Replacing s by ws for arbitrary $w \in S$ and using the above relation, we get

$$[w, r]U\phi(r) = \{0\}, \quad \text{for all } r \in U, w \in S.$$

This yields that

$$[w, r]SU\phi(r) = \{0\}, \quad \text{for all } r \in U, w \in S.$$

Since S is prime we have that for each $r \in U$, either $[w, r] = 0$ or $U\phi(r) = \{0\}$. The sets of r for which these conditions hold are additive subgroups of U with union equal to U ; hence either $U\phi(U) = \{0\}$ or $[w, r] = 0$ for all $r \in U$. If we apply Lemma 2.1 to the latter case, the ring turns to be commutative.

If $U\phi(U) = \{0\}$, then multiplying (1) from left by an arbitrary element $t \in U$ shows that

$$t\psi(r)\phi(s) = 0 \quad \text{for all } r, s, t \in U.$$

Replacing r by wr for arbitrary $w \in S$, and using $U\phi(U) = \{0\}$ we get $US\phi(r)\phi(s) = \{0\}$ for all $r, s \in U$. The primeness of S forces that $\phi(r)\phi(s) = 0$

for all $r, s \in U$, which is the desired conclusion that $\phi(U)\phi(U) = \{0\}$.

Now we turn to investigate the other case, that is $\phi(r) + \phi(\psi(r)) = 0$ for all $r \in U$. In (1), put $s = \psi(r)s$ and simplifying to get

$$[\psi(r), r]\phi(s) + \phi(r)[\psi(r), s] - \phi(r)[s, r] = -\phi([\psi(r), r])s, \quad \text{for all } r, s \in U. \quad (9)$$

Again in (1) we put $s = rs$, we get

$$[\psi(r), r]\phi(s) + \phi(r)[\psi(r), s] - \phi(r)[s, r] = 0, \quad \text{for all } r, s \in U. \quad (10)$$

Thus substituting from (9) in (10), we get $\phi([\psi(r), r])s = 0$ for all $r, s \in U$, that is,

$$\phi([\psi(r), r])U = \{0\} \quad \text{for all } r \in U.$$

But U is a left ideal, hence $\phi([\psi(r), r]) = 0$. This implies that

$$[\phi(\psi(r)), r] + [\psi(r), \phi(r)] = 0, \quad \text{for all } r \in U.$$

Using (1), the second summand in the above relation turns to be zero. And then using the fact that $\phi(\psi(r)) = -\phi(r)$, we get that $[\phi(r), r] = 0$ for all $r \in U$ and then Lemma 2.2 shows that the ring is commutative. This ends the proof of our theorem. $\square$

A slight modification in the proof of the last result yields the following.

Theorem 3.2. *Let S be a prime ring and U a nonzero left ideal. If S admits a nonzero generalized derivation ψ associated with a nonzero derivation ϕ satisfying $[\psi(r), \phi(s)] = \phi([r, s])$ for all $r, s \in U$, then S is commutative or $\phi(U)\phi(U) = U\phi(U) = \{0\}$.*

After we present the last two results, a natural question may be raised. What will happen if the generalized derivation is zero. Clearly, this will change our condition to be $\phi(rs) = \phi(sr)$ for each pair r, s in the nonzero left ideal U of the ring. Bell and Daif [1] showed by a counter example that in this case the ring will not be commutative. But fortunately, in the same paper they proved that if a prime ring S of characteristic different from 2 satisfying this condition, then S is commutative or $\phi^2(U) = \{0\} = \phi(U)\phi(U)$.

4 Open Problem

The open problem here is to get an answer to the question; Can we get the same results in this paper when ψ is an (α, β) -generalized derivation? that is, $\psi : S \rightarrow S$ be an additive mapping satisfies $\psi(rs) = \psi(r)\alpha(s) + \beta(r)\phi(s)$; where α and β are endomorphisms on S and ϕ is the related derivation to ψ .

References

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