

Study of Fuzzy D' -Co-implication with Respect to t-Norms and t-Conorms

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Abstract

The five most define of implication operators are $(T^*, \neg)$ -implications, QL -implications, residual implications, D -implication, and D' -implication. Also, fuzzy co-implication operators, generated from the t-norms, t-conorms and the fuzzy negations. The four most define of the co-implication operators are $((T^*, \neg)$ co-implication, residual co-implication, QL -co-implication, D -co-implication. In the current research, we will introduce the new definition of fuzzy D' -co-implication driven by the t-norm, t-conorm and the fuzzy negation and some properties and example of this fuzzy implication.

Keywords: Fuzzy implication, Fuzzy Co-implication, D' -co-implications.

1. Introduction

Fuzzy logic is known as a mathematical tool that was introduced by Lotfi A. Zadeh in 1965, (University of California, Berkeley) when he published his first celebrated paper opening the new field of Fuzzy Sets & Fuzzy Logic, and was applied to several fields, from control theory to artificial intelligence. Another extension of the fuzzy set is the set of interval-valued fuzzy (IVF) (Zadeh 1975), that can be defined by an interval-valued membership function. see [1] and [2].

Since characteristic functions are functional representations of sets, they play a crucial role in classical sets (also known as "crisp sets" in Fuzzy Set Theory). Their fuzzy counterparts are defined as membership functions, and their form is $\mu_A: X \rightarrow [0,1]$, where X is universe of discourse, i.e., μ_A , such that in the classical sets, introduces the set A and for every $x \in X$, $\mu_A(x)$ refers to the degree of possibility of x in A . see [3].

In the fuzzy set theory, relations are subsets of the Cartesian products, they are sets of form $A = \{(x, \mu_A(x)): x \in X\}$. see [3]. The basic connective in fuzzy logic are "AND" ($\wedge$), "OR" ($\vee$), "NOT" ($\neg$) are usually modeled as (t-norm T , t-conorm T^* and fuzzy negations $\neg$). The meaning of the conjunction $\wedge$ is triangular norm and disjunction $\vee$ mean is triangular conorm see [4], [5] and [6].

In fuzzy logic, implications are done by the suitable functions $I: [0,1] \times [0,1] \rightarrow [0,1]$, named implication operators, generated from the t-norms T , t-conorms T^* and

fuzzy negations $\neg$ (Fodor & Roubens, 1994). The five most usual ways to define these implication operators, for all $p, q \in [0,1]$, are:

1. $I(p, q) = T^*(\neg(p), q)$, called $(T^*, \neg)$ -Implications; (Baczynski & Jayaram, 2008)
2. $I(p, q) = \sup\{r \in [0,1]: T(r, p) \leq q\}$, for a given left-continuous t-norm T , called R-Implications; (Baczynski & Jayaram, 2008)
3. $I(p, q) = T^*(\neg(p), T(p, q))$, called QL-Implications; (Baczynski & Jayaram, 2010)
4. $I(p, q) = T^*(T(\neg(p), \neg(q)), q)$ called D -Implication. (Mas, Monserrat, Torrens, & Trillas, 2007)
5. $I(p, q) = \neg(T(T^*(p, q), \neg q))$ called D' -Implication. [7].

Fuzzy implications are very important in both the applied and theoretic communities of fuzzy set theory and was systematically studied by (Baczynski & Jayaram, 2008), (Bustince, Burillo, & Soria, 2003), [8]. While fuzzy co-implications have received little attention in the literature, and the well-studied classes are:

1. $(T, \neg)$ coimplications; (Jebril, 2016)
 $J_{T, \neg}(p, q) = T(q, \neg(p)), \forall p, q \in [0,1]$.
2. Residual complications; (Jebril, 2016)
 $J_{T^*}(p, q) = \inf\{r \in [0,1]: T^*(r, p) \geq q\}, \forall p, q \in [0,1]$.
3. QL-co-implications [9]
 $J_{T, T^*, \neg}(p, q) = T(T^*(p, q), \neg(p)), \forall p, q \in [0,1]$.
4. D -co-implications; (Jebril & Jablawi, 2019)
 $J^{T, T^*, \neg}(p, q) = T(T^*(\neg(p), \neg(q)), q), \forall p, q \in [0,1]$

In the current paper, we are about to define of D' -co-implications driven from t-norm, t-conorm and fuzzy negation. Also, some examples are also discussed as well as application as well. In addition, some new properties for D' -complication.

2. Triangular norms

Triangular norms that were introduced by K. Manger in (1942) when used in probabilistic metric spaces (PMS) to generalize the triangle inequality from the classical metric spaces (MS). The notion of the t-norm and its dual operation (t-conorm) is because of Schweizer and Sklar (1963). Both of t-norm and t-conorm could also be utilized as a generalization of the Boolean logic connectives to multi-valued logic. The conjunctive operator 'AND' generalize by t-norms and the disjunctive 'OR' operator generalize by the t-conorms. This situation allowed them to define the intersection and union operation in fuzzy logic, see [10].

$(T^*.1) T^*(p, q) = T^*(q, p);$	$(T^* \text{ is commutative})$
$(T^*.1) T^*(p, q) = T^*(q, p);$	$(T^* \text{ is commutative})$
$(T^*.3) T^*(p, T^*(q, r)) = T^*(T^*(p, q), r);$	$(T^* \text{ is associative})$
$(T^*.4) T^*(p, 0) = p.$	$(T^* \text{ has } 0 \text{ as identity})$

The conjunction $\wedge$ in fuzzy logic is usually modeled as follows:

Definition 2.2. [11] A function $T: [0,1] \times [0,1] \rightarrow [0,1]$ is a triangular norm (in short, t-norm), if for all $p, q, r, w \in [0,1]$ the following conditions are satisfied

Also, disjunction $\vee$ in fuzzy logic is often modeled as follows:

Definition 2.3. [11] A function $T^*: [0,1] \times [0,1] \rightarrow [0,1]$ is a triangular conorm (in short, t-conorm), if for all $p, q, r, w \in [0,1]$ the following conditions are satisfied

It is Lotfi Zadeh who first introduced a definition of the complement of the fuzzy set A of a universe of the discourse X as the fuzzy set A^c of X with the membership function $\mu_{A^c}: X \rightarrow [0,1]$ that is defined by $\mu_{A^c}(x) = \neg_c(\mu_A(x))$ for all $x \in X$, where the function $\neg_c: [0,1] \rightarrow [0,1]$, defined by $\neg_c(p) = 1 - p$ for all $p \in [0,1]$, can be seen as the fuzzy version of the classical negation, named standard fuzzy negation. Generally, we have:

Definition 2.4. (Fodor & Roubens, 1994) A function $\neg: [0,1] \rightarrow [0,1]$ is a negation function, if:

- (1) $\neg(0) = 1, \neg(1) = 0$;
- (2) $\neg(p) \leq \neg(q)$, if $p \geq q$. $\forall p, q \in [0,1]$. (Monotonicity)

In addition to this,

- A negation function is strict, if:
 - (1) $\neg(p)$ is continuous.
 - (2) $\neg(p) < \neg(q)$, if $p > q$. $\forall p, q \in [0,1]$.
- A negation function is strong or volutive, if:

$$\neg(\neg(p)) = p, \forall p \in [0,1].$$

- A negation function is weak, if $\neg$ is not strong.

Example 2.1. (Fodor & Roubens, 1994)

- The strong negation ($\neg_c(p) = 1 - p$),
- Strict negation but weak ($\neg_k(p) = 1 - p^2$),
- Weaker negation ($\neg_{D_1}(p) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$),
- Strongest negation ($\neg_{D_2}(p) = \begin{cases} 1 & \text{if } p < 1, \\ 0 & \text{if } p = 1. \end{cases}$).

Definition 2.5. [12] Let T is a t-norm and T^* is a t-conorm. The functions $\neg_T$ and $\neg_{T^*}$ from $[0,1]$ into $[0,1]$ that is defined by:

$$\neg_T(p) = \sup\{r \in [0,1] \mid T(p, r) = 0\}, \text{ for every } p \in [0,1],$$

$$\neg_{T^*}(p) = \inf\{r \in [0,1] \mid T^*(p, r) = 1\}, \text{ for every } p \in [0,1],$$

are called the natural negation of T and T^* , respectively.

The Law of Excluded Middle (*LEM*) is known as law of the well-known fundamental Boolean laws of the classical theory. When the *LEM*, in the classical logic, refers to that $\neg p \vee p$ is usually true. So that, we obtain the definition below. see (Massanet & Torrens, 2012).

Definition 2.6. (Bedregal, Cruz, & Santiago, 2012) Let T^* is a t-conorm and $\neg$ a fuzzy negation, the pair $(T^*, \neg)$ satisfies the *LEM* when

$$T^*(\neg(p), p) = 1, \forall p \in [0,1]. \quad (LEM)$$

Definition 2.7. (Bedregal, Cruz, & Santiago, 2012) Let T is a t-norm and $\neg$ a fuzzy negation, the pair $(T, \neg)$ fulfills the non-contradiction law when

$$T(\neg(p), p) = 0, \forall p \in [0,1]. \quad (2.1)$$

Proposition 2.1. [11] For the t-norm T , t-conorm T^* and strong negation $\neg$ so, T^* is $\neg$ -dual of T when

$$T^*(p, q) = \neg(T(\neg(p), \neg(q))), \forall p, q \in [0,1],$$

and T is $\neg$ -dual of S when

$$T(p, q) = \neg \left(T^*(\neg(p), \neg(q)) \right), \forall p, q \in [0, 1]. \quad (2.2)$$

Proposition 2.2. [5] Let T is a t-norm and T^* a t-conorm.

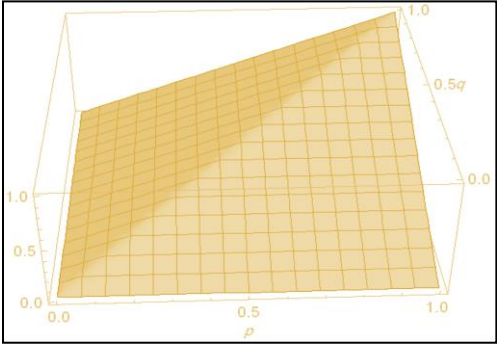
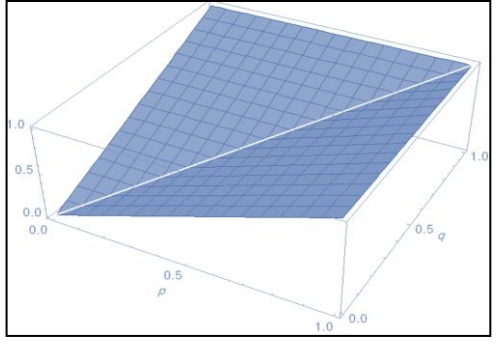
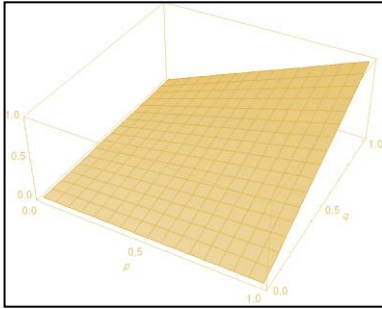
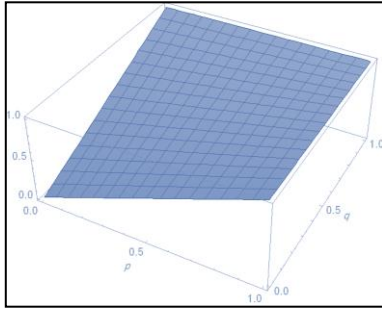
- (i) T^* is distributive over T iff $T = T_M$;
- (ii) T is distributive over S iff $T^* = T^*_M$;
- (iii) (T, T^*) is distributive pair iff $T = T_M$ and $T^* = T^*_M$.

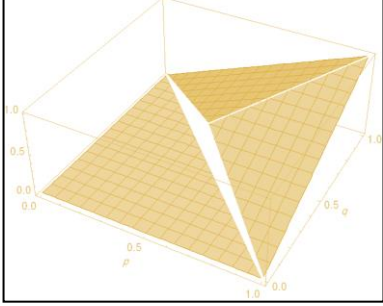
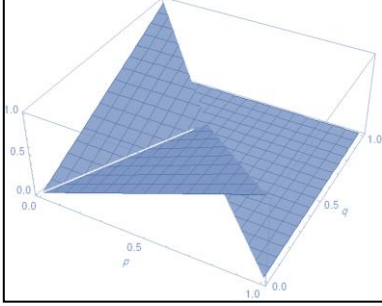
Proposition 2.3. [13] A function T^* is a t-conorm when there is a t-norm T such that

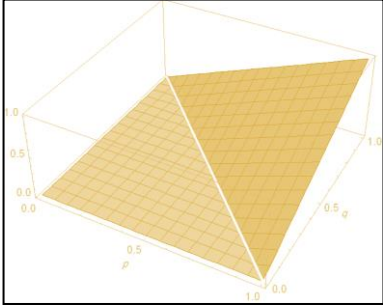
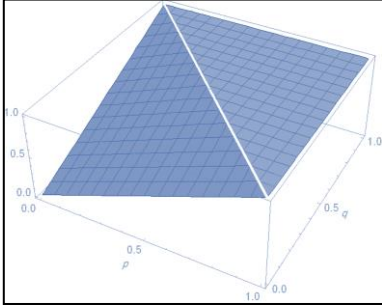
$$T^*(p, q) = 1 - T(1 - p, 1 - q), \forall p, q \in [0, 1].$$

In the current case, T^* is named the dual t-conorm of T .

Table 1.1 The standard examples of t-norm and N -dual of T (t-conorms).

norm (T)	t-conorm (T^*)
$T_M(p, q) = \min(p, q)$, (Minimum t-norm) 	$T^*_M(p, q) = \max(p, q)$, (Maximum t-conorm) 
Figure 9.2. $T_M(p, q)$	Figure 10.2. $T^*_M(p, q)$
norm (T)	t-conorm (T^*)
$\Pi(p, q) = pq$ (Product t-norm) 	$T^*_\Pi(p, q) = p + q - pq$, (Probabilistic sum t-conorm) 
Figure 11.2. $\Pi(p, q)$	Figure 12.2. $T^*_\Pi(p, q)$

norm (T)	t-conorm (T^*)
$N(p, q) = \begin{cases} \min(p, q) & \text{if } p + q \geq 1 \\ 0 & \text{if } p + q < 1. \end{cases}$ (Nilpotent t-norm)  Figure 13.2. $N(p, q)$	$T_N^*(p, q) = \begin{cases} \max(p, q) & \text{if } p + q < 1 \\ 0 & \text{if } p + q \geq 1. \end{cases}$ (Nilpotent t-conorm)  Figure 14.2. $T_N^*(p, q)$

t-norm (T)	t-conorm (T^*)
$L(p, q) = \max(p + q - 1, 0),$ (Lukasiewicz t-norm)  Figure 15.2. $L(p, q)$	$T_L^*(p, q) = \min(p + q, 1),$ (Bounded Sum t-conorm)  Figure 16.2. $T_L^*(p, q)$

t-norm (T)	t-conorm (T^*)
$W(p, q) = \begin{cases} p & \text{if } q = 1 \\ q & \text{if } p = 1 \\ 0 & \text{if } p, q \in [0, 1). \end{cases}$ (Drastic or weak t-norm)	$T_W^*(p, q) = \begin{cases} p & \text{if } q = 1 \\ q & \text{if } p = 1 \\ 1 & \text{otherwise.} \end{cases}$ (Drastic or largest t-conorm)

t-norm (T)	t-conorm (T^*)
$H(p, q) = \begin{cases} 0 & \text{if } p = q = 0 \\ \frac{pq}{p + q - pq} & \text{otherwise.} \end{cases}$ (Hamacher t-norm)	$T_H^*(p, q) = \begin{cases} 0 & \text{if } p = q = 0 \\ \frac{p + q - 2pq}{1 - pq} & \text{otherwise.} \end{cases}$ (Hamacher t-conorm)

t-norm (T)	t-conorm (T^*)
$D_\alpha(p, q) = \frac{pq}{\max(p, q, \alpha)}, \alpha \in (0, 1).$ (Dubois-Prade t-norm)	$T_{D_\alpha}^*(p, q) = 1 - \frac{(1-p)(1-q)}{\max(1-p, 1-q, \alpha)}, \alpha \in (0, 1).$ (Dubois-Prade t-conorm)

3. Fuzzy Co-implications

3.1. D' -implication

D -operations are the generalization of tautology $(p \Rightarrow q) \equiv [(\neg p \wedge \neg q) \vee q]$, and when the application of De Morgan's laws and the classical law of the double negation, we obtain the tautology below:

$$(p \Rightarrow q) \equiv ((\neg p \wedge \neg q) \vee q) \\ \equiv \neg((p \vee q) \wedge \neg q),$$

that leads us to a new family of fuzzy operations (respectively, implications) obtained by the formula below.

Definition 3.1.1 A function $I: [0,1]^2 \rightarrow [0,1]$ is named a D' -operation when there is a t-norm T , a t-conorm T^* , and a fuzzy negation $\neg$ such that ``

$$I(p, q) = \neg(T(T^*(p, q), \neg(q))), p, q \in [0,1].$$

If I is a D' -operation generated by the triple, $(T, T^*, \neg)$, then we denote it by $I^{\neg, T, T^*}$.

Theorem 3.2 Let $I^{\neg, T, T^*}$ be a D' -operation; then $I^{\neg, T, T^*}$ satisfies

1. J_2 ,
2. $I(0,0) = 0$,
3. $I(1,1) = 1$,
4. $I(1,0) = 0$,
5. $I(0,1) = 1$,
6. $I(p, 1) = 1$,
7. $I(1, q) = q$, and
8. if $\neg$ is a strong negation, then $I^{\neg, T, T^*}$ satisfies left neutrality property $I(1, q) = q$.
9. Furthermore, $\neg_{I^{\neg, T, T^*}} = \neg$, where

$$\neg_{I^{\neg, T, T^*}}(p) = I^{\neg, T, T^*}(p, 0), p \in [0,1]. \\ I^{\neg, T, T^*}(p, q) = \neg(T(T^*(p, q), \neg(q))) \\ = \max\{1 - p - q + pq, q\}.$$

3.2. Fuzzy complications classes

Fuzzy co-implications, or “co-implications” for short, which are defined as dual operations of the fuzzy implications if we use strong negation, are also one of the new connectives utilized in the fuzzy logic and the fuzzy inference as they are the generalization of the binary co-implications introduced in the classical logic [14]. We present, but not be limited to, the following work to demonstrate the above point:

Although the fuzzy co-implications could be utilized in a dual approach of fuzzy implications, they were less discussed in the literature. However, the co-implications also hane an important role in the classical logic, fuzzy logic and intuitionistic fuzzy logic. See also [2], (Cornelis, Deschrijver, & Kerre, 2004), [15], (Cornelis, Deschrijver, & Kerre, 2004) and [16]. In (Oh & Kandel, 1991), (Oh & Kandel, 1991) and (Jebril, 2016), the concept of fuzzy co-implication was presented as a new approach to the approximate reasoning of expert systems utilizing the equivalence relation for modus ponens of the inference in the fuzzy expert systems instead of the fuzzy implication. The algebraic properties of fuzzy $(T, \neg)$ co-implications and the residual co-implications are investigated in (Jebril, 2016), to provide a theoretical background for the approximate reasoning applications. see [17].

Extensions of fuzzy complications aren't similar in the classical ones, leading to distinct $((T, \neg)$ coimplications, QL coimplications, R coimplications) classes of fuzzy implications and additionally, there are distinct (explicit, implicit and axiomatic) representations of those connectives. See, e.g., (Baczynski & Jayaram, 2006), (Baczynski & Jayaram, 2008), [18] and [19].

While fuzzy implications are extensions of Boolean implication $\Rightarrow (p \Rightarrow q$ means that p is sufficient for q), as dualities, fuzzy co-implications are extensions of the Boolean co-implication $\Leftarrow (p \Leftarrow q$ means that p isn't necessary for q) [8]. Here, we first recall the definition of fuzzy co-implications presented by De Baets in [2].

Definition 3.1.2 [2] A function $J: [0,1] \times [0,1] \rightarrow [0,1]$ is a fuzzy coimplication if, $\forall p, q, r \in [0,1]$, the following conditions are fulfilled:

$$(J1) J(1,1) = J(1,0) = J(0,0) = 0 \text{ and } J(0,1) = 1.$$

$$(J2) J(p, q) \geq J(r, q) \text{ if } p \leq r. (J \text{ is decreasing according to the first variable}).$$

$$(J3) J(p, q) \leq J(p, r) \text{ if } q \leq r. (J \text{ is increasing according to the second variable}).$$

The set of all fuzzy co-implication is denoted by $Co - FI$. From the previous definition, we are able to deduce that for every fuzzy coimplication $J(1, q) = J(p, 0) = 0$, $\forall p, q \in [0,1]$. Moreover, J fulfills also the normality condition $J(p, p) = 0$.

There exists also a property which relates fuzzy co-implications and negations:

$$(J4) \text{ Contraposition: } J(p, q) = J(\neg(q), \neg(p)), \text{ for all } p, q \in [0,1];$$

$$(J4a) \text{ Left contraposition: } J(J(p, q), J(\neg(q), \neg(p))) = 0, \text{ for all } p, q \in [0,1];$$

$$(J4b) \text{ Right contraposition: } J(J(\neg(q), \neg(p)), J(p, q)) = 0, \text{ for all } p, q \in [0,1].$$

Remark 3.3. [2] Directly from Definition 3.12 we observe that every fuzzy co-implication J fulfills the left and right boundary conditions mentioned below, respectively:

$$J(1, q) = 0, q \in [0, 1], \quad (\text{Co-LB})$$

$$J(p, 0) = 0, p \in [0, 1]. \quad (\text{Co-RB})$$

Therefore, J also fulfills the normality condition:

$$J(1, 0) = 0. \quad (\text{Co-NC})$$

Lemma 3.1 (Jebril, 2016) when a function $J: [0,1] \times [0,1] \rightarrow [0,1]$ fulfills (J1) and (J2), the function $\neg_J: [0,1] \rightarrow [0,1]$ defined by

$$\neg_J(p) = J(p, 1), \forall p \in [0,1],$$

is a fuzzy negation.

The properties below are the generalization of the fuzzy co-implication from the classical logic.

Definition 3.1.3 [20] A fuzzy co-implications J is said to fulfill the following most important properties, $\forall p, q, r \in [0,1]$.

The left neutrality property,

$$J(0, q) = q; \quad (\text{Co-NP})$$

Identity principle,

$$J(p, p) = 0; \quad (\text{Co-IP})$$

The exchange principle,

$$J(p, J(q, r)) = J(q, J(p, r)); \quad (\text{Co-EP})$$

The ordering property,

$$J(p, q) = 0 \Leftrightarrow p \geq q. \quad (\text{Co-OP})$$

There exist 4 best-known classes of the fuzzy co-implications, $(T, \neg)$, residual, QL and D co-implication. In 2016, Jebril, I. utilized the idea of the function of fuzzy co-implication by presenting the definition of $(T, \neg)$ and residual co-implication. Let's start with the first type and it is $(T, \neg)$ co-implication.

3.3.1. $(T, \neg)$ co-implication

All definitions, theorems, and examples in section in the following taken from (Jebril, 2016).

A $(T, \neg)$ coimplication is generated from conjunction ' $\wedge$ ' and Boolean negation ' $\neg$ ', obtaining idea from the equivalency in the classical logic.

$$q \neq p \equiv q \wedge \neg p, \forall p, q \in [0, 1].$$

Definition 3.1.4 A function $J: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is named a $(T, \neg)$ co-implication when there is a t-norm T and a fuzzy negation $\neg$ such that

$$J_{T, \neg}(p, q) = T(q, \neg(p)), \forall p, q \in [0, 1]. \quad (3.2)$$

The $(T, \neg)$ coimplication driven by a t-norm T and a fuzzy negation $\neg$ is denoted by $J_{T, \neg}$.

Proposition below states how an $(T^*, \neg)$ implication gives rise to a fuzzy $(T, \neg)$ co-implication and vice-versa.

Proposition 3.2. A function $J_{T, \neg}$ from $[0, 1]^2$ into $[0, 1]$ is a $(T, \neg)$ coimplication with strong negation if

$$J_{T, \neg}(p, q) = \neg(I_{T^*, \neg}(q, p)), \quad (3.3)$$

for some $I_{T^*, \neg}$ and fuzzy (strong) negation $\neg$.

Conversely, $I_{T^*, \neg}$ from $[0, 1]^2$ into $[0, 1]$ is an $(T^*, \neg)$ implication with strong negation if

$$I_{T^*, \neg}(p, q) = \neg(J_{T, \neg}(q, p)),$$

for some $J_{T, \neg}$ and fuzzy (strong) negation $\neg$.

Theorem 3.3 The following statements are true:

- (1) From t-norm T , then $J_{T, \neg} \in Co - FI$.
- (2) All $(T, \neg)$ co-implication are fuzzy co-implications satisfy (Co-NP) and (Co-EP).
- (3) If $I_{T^*, \neg}$ fulfills (IP) with strong negation, then $J_{T, \neg}$ satisfies (Co-IP).
- (4) If $I_{T^*, \neg}$ fulfills (OP) with strong negation, then $J_{T, \neg}$ satisfies (Co-OP).

Theorem 3.4 For t-norm T and a fuzzy negation $\neg$, then

$$J_{T, \neg}(p, p) = 0, \text{ for all } p \in [0, 1],$$

if

$$T(p, \neg(p)) = 0, \text{ for all } p \in [0, 1].$$

Example 3.7. Table 5.3 lists $(T, \neg)$ coimplication obtained from the basic t-norms and fuzzy negations.

Table 5.3. Examples of some basic $(T, \neg)$ Co-implications.

T	$\neg$	$J_{T, \neg}$
T_M	$\neg_c$	$J_{T_M, \neg_c}(p, q) = \min(q, 1 - p)$
Π	$\neg_c$	$J_{\Pi, \neg_c}(p, q) = q - pq$

3.3.2. Residual Co-implication

The dual of Heyting algebra is named Brouwerian algebra $(L, \wedge, \vee, \overset{*}{\rightarrow}, 0, 1)$ is a lattice with 0 and 1, and the binary operation named coimplication $\overset{*}{\rightarrow}$ in dual Heyting algebra. Satisfying $\forall p, q, r \in L$.

$$p \vee r \geq q \leftrightarrow p \rightarrow^* q.$$

The set of all $r \in L$ such that $p \vee r \geq q$ involves the smallest element, denoted by $p \rightarrow^* q$. Precisely

$$p \rightarrow^* q = \inf\{r \in L: p \vee r \geq q\}.$$

Definition 3.1.5 Let T^* be the t-conorm of the right continuous T . Then, the residual coimplication (R^* -implication) obtained from T^* , is

$$J_{T^*}(p, q) = \inf\{r \in [0,1]: T^*(r, p) \geq q\}, \forall p, q \in [0,1].$$

The R^* -implication driven by the t-conorm T^* is denoted by J_{T^*} .

3.3.3. QL-co-implication

Then in 2017, Ghoneim, H. and Jebril, I. studied the class of QL-co-implication under defined conditions in crisp logic and fuzzy logic in [9]. And all definitions, theorems and examples in this section taken from [9].

A QL-co-implication is generated from a fuzzy negation, a t-norm and a t-conorm, getting idea from the equivalency in the classical logic.

$$q \not\Rightarrow p \equiv ((p \vee q) \wedge \neg p), \forall p, q \in [0,1].$$

Definition 3.1.6 A function $J: [0,1] \times [0,1] \rightarrow [0,1]$ is called a QL-co-implication when there is a t-norm T , t-conorm T^* and a fuzzy negation $\neg$ such that:

$$J_{T,T^*,\neg}(p, q) = T(T^*(p, q), \neg(p)), \forall p, q \in [0,1].$$

The QL-co-implication generated by a t-norm T , a t-conorm T^* and a fuzzy negation $\neg$ is denoted by $J_{T,T^*,\neg}$.

Proposition 3.3 Let $J_{T,T^*,\neg}$ be a QL-co-implication, then $\neg_{J_{T,T^*,\neg}} = \neg$.

Example 3.9. Table 7.3 lists QL-co-implication obtained from any t-norm T and t-conorm T^* and a fuzzy negation $\neg$.

Table 7.3. Examples of some basic QL-co-implications.

T	T^*	$\neg$	$J_{T,T^*,\neg}$	Name
T_M	T_M^*	$\neg_c$	$J_{ZD}(p, q) = \min(\max(p, q), 1 - p)$	Early Zadeh
Π	T_{Π}^*	$\neg_c$	$J_{\Pi}(p, q) = p(1 - 2q - p + pq) + q$	Klir and Yuan

We have introduced in this thesis some new properties for QL-coimplication and are associated with D coimplication.

Lemma 3.2. Every QL-coimplication satisfies (J1), (J3), (Co-LB) and (Co-NP).

Proof. Let $J_{T,T^*,\neg}$ be a QL-coimplication, so $J_{T,T^*,\neg}$ satisfies (J1) since:

$$J_{T,T^*,\neg}(1,1) = J_{T,T^*,\neg}(1,0) = J_{T,T^*,\neg}(0,0) = 0,$$

and

$$J_{T,T^*,\neg}(0,1) = T(T^*(0,1), \neg(0)) = T(T^*(0,1), 1) = 1.$$

Now, assume that $p, q, r \in [0,1]$ and $q \leq r$. Then, $T^*(p, q) \leq T^*(p, r)$.

$$\Rightarrow T(T^*(p, q), \neg(p)) \leq T(T^*(p, r), \neg(p))$$

$$\Rightarrow J_{T,T^*,\neg}(p, q) \leq J_{T,T^*,\neg}(p, r).$$

Hence $J_{T,T^*,\neg}$ satisfies (J3).

Let $J_{T,T^*,\neg}$ be a QL-coimplication, for all $q \in [0,1]$.

$$J_{T,T^*,\neg}(1, q) = T(T^*(1, q), \neg(1)) = T(1, 0) = 0,$$

Then $J_{T,T^*,\neg}$ satisfies (Co-LB).

For any $q \in [0,1]$,

$$J_{T,T^*,\neg}(0, q) = T(T^*(0, q), \neg(0)) = T(T^*(0, q), 1) = T^*(0, q) = q.$$

Hence $J_{T,T^*,\neg}$ satisfies (Co-NP).

Lemma 3.3 For the t-norm T , t-conorm T^* and strong fuzzy negation $\neg$. If QL-coimplication satisfies (Co-EP), then it also satisfies (J4).

Proof. Let $J_{T,T^*,\neg}$ satisfies (Co-EP), for all $p, q \in [0,1]$.

$$\begin{aligned} J_{T,T^*,\neg}(\neg(q), \neg(p)) &= J_{T,T^*,\neg}(\neg(q), J_{T,T^*,\neg}(p, 1)) \\ &= J_{T,T^*,\neg}(p, J_{T,T^*,\neg}(\neg(q), 1)) \end{aligned}$$

$$= J_{T,T^*,\neg}(p, q).$$

3.3.4. D-co-implication

In the classical logic, the operator ' $\nRightarrow$ ' is driven by Boolean negation ' $\neg$ ', conjunction ' $\wedge$ ' and disjunction ' $\vee$ ':

$$q \nRightarrow p \equiv (\neg p \vee \neg q) \wedge q, \forall p, q \in [0,1].$$

Definition 4.2.1. A function $J: [0,1] \times [0,1] \rightarrow [0,1]$ is named a D co-implication when there is a t-norm T , a t-conorm S and fuzzy negation N such that

$$J^{T,S,N}(p, q) = T(S(N(p), N(q)), q), \text{ for all } p, q \in [0,1].$$

If N is a strong negation, J is named a strong D co-implication. If N isn't strong negation, then J is named non-strong D-co-implication. Moreover, if a D-co-implication is generated from T , S and N , then we will denote this by $J^{T,S,N}$.

Lemma 3.4. Every D co-implication fulfills (J1), (J2), (Co-RB) and (Co-NP).

Remark 3.4. Given any D co-implication $J^{T,S,N}$ it is trivially satisfied that $J^{T,S,N}(p, 1) = N(p)$. So, if any D co-implication fulfills the (Co-EP) then it also fulfills (J4), since

$$\begin{aligned} J^{T,S,N}(N(q), N(p)) &= J^{T,S,N}(N(q), J^{T,S,N}(p, 1)) \\ &= J^{T,S,N}(p, J^{T,S,N}(N(q), 1)) \\ &= J^{T,S,N}(p, q). \end{aligned}$$

■

Remark 3.5. For t-norm T , t-conorm S and a fuzzy negation N , Then

$$J^{T,S,N}(p, p) = 0, \forall p \in [0,1],$$

if and only if

$$T(N(p), p) = 0, \forall p \in [0,1].$$

Example 3.10. For a t-norm T_M , t-conorm S_M and a strong fuzzy negation N_c then D co-implication J^{T_M, S_M, N_c} given by

$$\begin{aligned} J^{T_M, S_M, N_c}(p, q) &= T_M(S_M(N_c(p), N_c(q)), q) \\ &= T_M(S_M(1-p, 1-q), q) \\ &= T_M(\max(1-p, 1-q), q) \\ &= \min(\max(1-p, 1-q), q). \end{aligned}$$

4. Fuzzy D' -Co-implications

4.1. Introduction

The co-implications are in general performed using suitable functions $J: [0,1] \times [0,1] \rightarrow [0,1]$, named co-implication operators, generated from t-norms, t-conorms and fuzzy negations.

From chapter three, for each p and q in $[0,1]$, the 4 most usual ways to define those co-implication operators are:

1. $J_{T,\neg}(p, q) = T(q, \neg(p))$ for a given t-norm T and a fuzzy negation $\neg$, called $(T, \neg)$ co-implications;
2. $J_{T^*}(p, q) = \inf\{r \in [0,1]: T^*(r, p) \geq q\}$ for a given right-continuous t-norm T , named residual co-implications;
3. $J_{T,T^*,\neg}(p, q) = T(T^*(p, q), \neg(p))$ for a given t-norm T , a t-conorm T^* and a fuzzy negation $\neg$, named QL co-implications.

p	q	$p \Rightarrow q$	$q \nRightarrow p$	$\neg q$	$p \wedge q$	$(p \wedge q) \vee \neg q$	$\neg((p \wedge q) \vee \neg q)$
1	1	1	0	0	1	1	0
1	0	0	0	1	0	1	0
0	1	1	1	0	0	0	1
0	0	1	0	1	0	1	0

4. $J^{T, T^*, \neg}(p, q) = T(T^*(\neg(p), \neg(q)), q)$ for a given t-norm T , a t-conorm T^* and a fuzzy negation $\neg$, that we will name D -co-implication.

we mentioned that in 2016, Jebril, I., uses the idea of the function of fuzzy co-implication by introduce the definition of $(T, \neg)$ and residual co-implication (Jebril, I., 2016). In 2017, Ghoneim, H. and Jebril, I. investigated the class of QL -co-implication under defined conditions in the crisp logic and fuzzy logic in [9]. In 2019, Jebril, I. and Jablawi, E. investigated some Properties of fuzzy D -Co-implication (Jebril & Jablawi, 2019).

In this chapter, we will conclude the fifth definition of the fifth most usual ways to define these co-implication operators, the so-called D' -co-implication. Also, some characteristics are discussed and also some examples as well. Also, study the equivalences between fuzzy D' -co-implication with fuzzy co-implication classes.

4.2. D' -Co-implications

Here, we will study the fifth type of fuzzy D' -co-implications, that is named D' -co-implication.

In classical logic, the operator ' $\nRightarrow$ ' is generated by Boolean negation ' $\neg$ ', conjunction ' $\wedge$ ' and disjunction ' $\vee$ ':

$$q \nRightarrow p \equiv \neg((p \wedge q) \vee \neg q), \forall p, q \in [0, 1].$$

In the following table 1.4, we can see the truth table for the classical D' -co-implication.

Table 1.4. D' -co-implication

After we concluded the previous relationship in the classical logic and verified its validity by using the truth table, we will define the D' -co-implication by replacing the negation with the negation, the disjunction with T^* , and the conjunction with T as follows:

Definition 4.1. A function $J: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a D' -co-implication if there exists a fuzzy negation $\neg$, a t-conorm T^* and a t-norm T such that:

$$J^{\neg, T^*, T}(p, q) = \neg(T^*(T(p, q), \neg(q))), \text{ for all } p, q \in [0, 1].$$

If $\neg$ is a strong negation, then J is called a strong D' -co-implication. If $\neg$ is not strong negation, then J is called non-strong D' -co-implication.

Moreover, if a D' -co-implication is generated from $\neg, T^*$ and T , then we will denote this by $J^{\neg, T^*, T}$.

Lemma 4.1. Every D' -co-implication satisfies (J_1) , (J_2) , $(Co - RB)$ and $(Co - Np)$.

Proof. Let $J^{\neg, T^*, T}$ is a D' co-implication, so $J^{\neg, T^*, T}$ satisfies (J_1) since

$$J^{\neg, T^*, T}(1, 1) = J^{\neg, T^*, T}(1, 0) = J^{\neg, T^*, T}(0, 0) = 0,$$

$$\text{and } J^{\neg, T^*, T}(0, 1) = \neg(T^*(T(0, 1), \neg(1))) = \neg(T^*(0, 0)) = \neg(0) = 1.$$

Now, assume that $p_1, p_2, q \in [0, 1]$ and $p_1 \leq p_2$, Then

$$\begin{aligned} \neg(p_1) &\geq \neg(p_2) \\ T^*(T(p_1, q), \neg(q)) &\leq T^*(T(p_2, q), \neg(q)) \\ \neg(T^*(T(p_1, q), \neg(q))) &\geq \neg(T^*(T(p_2, q), \neg(q))) \end{aligned}$$

Hence $J^{\neg, T^*, T}$ satisfies (J_2) .

Let $J^{\neg, T^*, T}$ be a D' co-implication, for all $p \in [0, 1]$.

$$\begin{aligned} J^{\neg, T^*, T}(p, 0) &= \neg(T^*(T(p, q), \neg(q))) \\ &= \neg(T^*(T(p, 0), \neg(0))) \\ &= \neg(T^*(0, 1)) = \neg(1) = 0 \text{ (By positives)} \end{aligned}$$

Then $J^{\neg, T^*, T}$ satisfies $(Co - RB)$, for any $q \in [0, 1]$

$$J^{\neg, T^*, T}(0, q) = \neg(T^*(T(0, q), \neg(q)))$$

and $T(0, q) = 0$ (by the ordering on all t-norms).

Since $\neg(T^*(0, \neg(q))) = \neg(\neg(q)) = q$

$J^{\neg, T^*, T}(0, q) = q$. Hence $J^{\neg, T^*, T}$ satisfies $(Co - Np)$.

Remark 4.1. given any D' -co-implication, $J^{\neg, T^*, T}$ it is trivially satisfied that

$$J^{\neg, T^*, T}(p, 1) = \neg(p),$$

Thus, when any D' co-implication fulfills, the $(Co - Ep)$, then it also satisfies (J_4) since

$$\begin{aligned} J^{\neg, T^*, T}(\neg(q), \neg(p)) &= J^{\neg, T^*, T}(\neg(q), J^{\neg, T^*, T}(p, 1)) \\ &= J^{\neg, T^*, T}(p, J^{\neg, T^*, T}(\neg(q), 1)) \\ &= J^{\neg, T^*, T}(p, q). \end{aligned}$$

Proposition 4.1. let $J^{\neg, T^*, T}$ be a D' co-implication, then

$$\neg_{J^{\neg, T^*, T}} = \neg$$

Proof. For any $p \in [0, 1]$, we have

$$\begin{aligned} \neg_{J^{\neg, T^*, T}}(p) &= J^{\neg, T^*, T}(p, 1) = \neg(T^*(T(p, 1), \neg(1))) \\ &= \neg(T^*(p, 0)) = \neg(p). \end{aligned}$$

Example 4.1. For a t -conorm T^*_M , t -norm T_M and strong fuzzy negation $\neg_C$, then D' -co-implication $J^{\neg_C, T^*_M, T_M}$ given by

$$\begin{aligned} J^{\neg_C, T^*_M, T_M} &= \neg_C(T^*_M(T_M(p, q), \neg(q))) \\ &= \neg_C(T^*_M(\min(p, q), 1 - q)) \\ &= \neg_C(\max(\min(p, q), 1 - q)) \\ &= 1 - (\max(\min(p, q), 1 - q)). \end{aligned}$$

Example 4.2. for a t -norm Π , t -conorm T^*_Π and strong fuzzy negation $\neg_C$, then D' - co-implication

$$\begin{aligned} J^{\neg_C, T^*_\Pi, \Pi}(p, q) &= \neg_C(T^*_\Pi(\Pi(p, q), \neg(q))) = \neg_C(T^*_\Pi(p, q, 1 - q)) \\ &= \neg_C(p, q + 1 - q - ((p, q)(1 - q))) \\ &= 1 - (p, q + 1 - q - p, q + p, q^2). \end{aligned}$$

Example 4.3. for a strong fuzzy negation $\neg_C$, t -conorm T^*_L and t -norm L , then D' -co-implication $J^{\neg_C, T^*_L, L}$ given by

$$\begin{aligned} J^{\neg_C, T^*_L, L}(p, q) &= \neg_C(T^*_L(L(p, q), \neg(q))) \\ &= \neg_C(T^*_L(\max(p + q - 1), 0), \neg(q)) \\ &= \neg_C(T^*_L(\max(p + q - 1), 0), 1 - q) \\ &= \neg_C((\min((\max(p + q - 1), 0) + (1 - q), 1))) \\ &= 1 - (\min((\max(p + q - 1), 0) + (1 - q), 1))). \end{aligned}$$

Example 4.4. for a t -norm T_M , t -conorm T_N^* and strong fuzzy negation $\neg_C$, then D' co-implication $J^{\neg_C, T_N^*, T_M}$

$$J^{\neg_C, T_N^*, T_M}(p, q) = \neg_C \left(T_N^*(T_M(p, q), \neg(q)) \right)$$

Example 4.5. For a t -norm T_M , t -conorm T_M^* and strict negation $\neg_K$ but not strong, then D' co-implication $J^{\neg_K, T_M^*, T_M}$ given by

$$\begin{aligned} & 1 - \left(\max(\min(p, q), 1 - q^2) \right)^2 \\ &= \neg_C \left(T_N^*(\min(p, q), \neg(q)) \right) \\ &= \neg_C \left(T_N^*(\min(p, q), 1 - q) \right) \\ &= \neg_C \begin{cases} \max(\min(p, q), 1 - q), & \text{if } \min(p, q) + 1 - q < 1 \\ 0, & \text{if } \min(p, q) + 1 - q \geq 1 \end{cases} \\ &= 1 - \begin{cases} \max(\min(p, q), 1 - q), & \text{if } \min(p, q) + 1 - q < 1 \\ 0, & \text{if } \min(p, q) + 1 - q \geq 1. \end{cases} \end{aligned}$$

5. Conclusion and future work

In this research we have studied the class of D' -co-implication under certain conditions in crisp logic and fuzzy logic. We has completed the study of the four usual models of fuzzy coimplications that a $(T, \neg)$ co-implication, residual co-implication, QL -co-implication, and D -co-implication. We notice that, throughout this research D' -co-implications are defined from fuzzy negations, and in the aim to maintain the similarity to D -co-implication definition, we did not require that D' -co-implication underlying negation must be strong. By drawing for examples and some properties of D' -co-implication we conclude that the definition of D' -co-implication doesn't depend on strong negation. Also, D' -co-implications defined from non-strong negation aren't contraposition of D' -co-implications. These types of fuzzy co-implications are very exciting to learn more about their characteristics.

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