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Corrections to Several Integral Inequalities of Hermite–Hadamard Type for s -Geometrically Convex Functions

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Abstract

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1 Motivations

The concept of s -convex mappings was introduced in [4, Definition 1.9] as follows.

Definition 1.1 ([4, Definition 1.9]). *For some $s \in (0, 1]$, a function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}_+$ is said to be an s -geometrically convex function if the inequality*

$$f(x^\lambda y^{1-\lambda}) \leq [f(x)]^{\lambda^s} [f(y)]^{(1-\lambda)^s} \quad (1)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Remark 1.2. *Let $s \in (0, 1]$ and let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an s -geometrically convex function.*

1. *If $s = 1$, the s -geometric convexity becomes a geometric convexity on $\mathbb{R}_+$;*
2. *If $s \in (0, 1)$, then $f(x) \geq 1$ for all $x \in I$. This can be deduced immediately by taking $x = y$ in (1).*

The following integral identity was created in the paper [2].

Lemma 1.3 ([2, Lemma 2.1]). *Let $I \subseteq \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be differentiable on I° , the interior of I , and $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b) dt.$$

In [3], using Lemma 1.3, the following Hermite–Hadamard inequalities about s -geometrically convex functions are established.

Theorem 1.4 ([3, Theorem 2.1]). *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the points $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'(a)| \leq 1$ and $|f'(x)|$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $s \in (0, 1]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} G_1(s; g_1(\alpha), g_2(\alpha)),$$

where

$$\begin{aligned} g_1(\alpha) &= \begin{cases} \frac{1}{4}, & \alpha = 1; \\ \frac{2\alpha^{1/2} - 2 - \ln \alpha}{(\ln \alpha)^2}, & \alpha \neq 1, \end{cases} \\ g_2(\alpha) &= \begin{cases} \frac{1}{4}, & \alpha = 1; \\ \frac{2\alpha^{1/2} - 2\alpha + \alpha \ln \alpha}{(\ln \alpha)^2}, & \alpha \neq 1, \end{cases} \\ \alpha(u, v) &= |f'(a)|^u |f'(b)|^v, \quad u, v > 0, \end{aligned} \quad (2)$$

and

$$G_1(s; g_1(\alpha), g_2(\alpha)) = |f'(b)|^s [g_1(\alpha(s, s)) + g_2(\alpha(s, s))].$$

Theorem 1.5 ([3, Theorem 2.2]). *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the numbers $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $q > 1$, if $|f'(a)| \leq 1$, and if $\frac{1}{p} + \frac{1}{q} = 1$ and $s \in (0, 1]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} G_2(s, q; g_3(\alpha)),$$

where

$$g_3(\alpha) = \begin{cases} 1, & \alpha = 1; \\ \frac{\alpha - 1}{\ln \alpha}, & \alpha \neq 1 \end{cases}$$

and

$$G_2(s, q; g_3(\alpha)) = |f'(b)|^s [g_3(\alpha(qs, qs))]^{1/q}.$$

Theorem 1.6 ([3, Theorem 2.4]). *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the points $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'(a)| \leq 1$ and $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$ and $s \in (0, 1]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} G_3(s, q; g_1(\alpha), g_2(\alpha)),$$

where $g_1(\alpha)$ and $g_2(\alpha)$ are defined in (2), and

$$G_3(s, q; g_1(\alpha), g_2(\alpha)) = |f'(b)|^s \{ [g_1(\alpha(qs, qs))]^{1/q} + [g_2(\alpha(qs, qs))]^{1/q} \}.$$

Remark 1.7. *From the conditions in Theorems 1.4 to 1.6, we derive that $|f'(x)| \leq 1$ for $x \in [a, b]$.*

Let $s \in (0, 1]$, $q \geq 1$, and $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$. By the second item in Remark 1.2, under conditions of Theorems 1.4 to 1.6,

1. *if $s = 1$, all the results in Theorems 1.4 to 1.6 hold;*
2. *if $s \in (0, 1)$ and $|f'(x)| = 1$ almost everywhere on I , the results in Theorems 1.4 to 1.6 hold on I .*

In this paper, we aim to correct errors appeared in Theorems 1.4 to 1.6.

2 Lemmas

In this paper, we will apply the following lemmas.

Lemma 2.1 ([1, p. 4]). *Let $0 < \mu \leq 1 \leq \eta$ and $0 < s, t \leq 1$. Then*

$$\mu^{ts} \leq \mu^{st} \quad \text{and} \quad \eta^{ts} \leq \eta^{st+1-s}.$$

Lemma 2.2. *Let $\alpha \in (0, 1]$. Then*

$$M(\alpha) \triangleq \int_0^1 |1 - 2t| \alpha^t \, dt = \begin{cases} \frac{1}{2}, & \alpha = 1; \\ \frac{(\alpha - 1) \ln \alpha - 2(\sqrt{\alpha} - 1)^2}{[\ln \alpha]^2}, & \alpha \neq 1 \end{cases} \quad (3)$$

and

$$N(\alpha) \triangleq \int_0^1 \alpha^t \, dt = \begin{cases} 1, & \alpha = 1; \\ \frac{\alpha - 1}{\ln \alpha}, & \alpha \neq 1. \end{cases} \quad (4)$$

Proof. It is straightforward. □

3 Corrections

Now we begin to correct errors in Theorems 1.4 and 1.6 as Theorem 3.1 below.

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the numbers $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $s \in (0, 1]$ and $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$, then*

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\ \leq \frac{b - a}{2^{2-1/q}} |f'(a)|^{1-s} |f'(b)| [M(|f'(a)| |f'(b)|^{-1})^{sq}]^{1/q}, \end{aligned}$$

where $M(\alpha)$ is defined in (3).

Proof. Using Lemma 1.3 and by Hölder's integral inequality, we have

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| &\leq \frac{b - a}{2} \int_0^1 |1 - 2t| |f'(ta + (1 - t)b)| \, dt \\ &\leq \frac{b - a}{2} \left(\int_0^1 |1 - 2t| \, dt \right)^{1-1/q} \left[\int_0^1 |1 - 2t| |f'(ta + (1 - t)b)|^q \, dt \right]^{1/q}. \quad (5) \end{aligned}$$

By the s -geometric convexity and monotonicity of $|f'|^q$ on $[a, b]$ and by virtue of Lemma 2.1, we obtain

$$\begin{aligned} |f'(ta + (1-t)b)|^q &\leq |f'(a^t b^{1-t})|^q \leq |f'(a)|^{qt^s} |f'(b)|^{q(1-t)^s} \\ &\leq |f'(a)|^{q(1-s)} |f'(b)|^q [|f'(a)|^{sq} |f'(b)|^{-sq}]^t. \end{aligned} \quad (6)$$

Substituting the inequality (6) into the inequality (5) and by Lemma 2.2, we arrive at

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{b-a}{2} \int_0^1 |1-2t| |f'(ta + (1-t)b)| \, dt \\ &\leq \frac{b-a}{2} \left(\int_0^1 |1-2t| \, dt \right)^{1-1/q} \left[\int_0^1 |1-2t| |f'(ta + (1-t)b)|^q \, dt \right]^{1/q} \\ &\leq \frac{b-a}{2} \left(\frac{1}{2} \right)^{1-1/q} \left[|f'(a)|^{q(1-s)} |f'(b)|^q \int_0^1 |1-2t| [|f'(a)|^{sq} |f'(b)|^{-sq}]^t \, dt \right]^{1/q} \\ &= \frac{b-a}{2} \left(\frac{1}{2} \right)^{1-1/q} |f'(a)|^{1-s} |f'(b)| [M(|f'(a)||f'(b)|^{-1})^{sq}]^{1/q}. \end{aligned}$$

Theorem 3.1 is thus proved. $\square$

If taking $q = 1$ in Theorem 3.1, we can derive the following corollary.

Corollary 3.2. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the points $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $s \in (0, 1]$ and $|f'(x)|$ is s -geometrically convex and monotonically decreasing on $[a, b]$, then*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ &\leq \frac{b-a}{2} |f'(a)|^{1-s} |f'(b)| M(|f'(a)||f'(b)|^{-1})^s, \end{aligned}$$

where $M(\alpha)$ is defined in (3).

If letting $s = 1$ in Theorem 3.1, we obtain

Corollary 3.3. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the numbers $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'(x)|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2^{2-1/q}} |f'(b)| [M(|f'(a)||f'(b)|^{-1})^q]^{1/q},$$

where $M(\alpha)$ is defined in (3).

Now we correct errors appeared in Theorem 1.5 as follows.

Theorem 3.4. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the points $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $s \in (0, 1]$ and $|f'(x)|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $q > 1$ with $\frac{1}{q} + \frac{1}{p} = 1$, then*

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{2(p+1)^{1/p}} |f'(a)|^{1-s} |f'(b)| [N(|f'(a)||f'(b)|^{-1})^{sq}]^{1/q}, \end{aligned}$$

where $N(\alpha)$ is defined in (4).

Proof. Since $|f'|^q$ is s -geometrically convex and monotonically decreasing on $[a, b]$, by Lemmas 1.3 to 2.2 and by Hölder's inequality, we see that

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \int_0^1 |1-2t| |f'(ta + (1-t)b)| dt \\ & \leq \frac{b-a}{2} \left\{ \left(\int_0^1 |1-2t|^p dt \right)^{1/p} \left[\int_0^1 |f'(ta + (1-t)b)|^q dt \right]^{1/q} \right\} \\ & \leq \frac{b-a}{2(p+1)^{1/p}} \left[|f'(a)|^{q(1-s)} |f'(b)|^q \int_0^1 [|f'(a)|^{sq} |f'(b)|^{-sq}]^t dt \right]^{1/q} \\ & = \frac{b-a}{2(p+1)^{1/p}} |f'(a)|^{1-s} |f'(b)| [N(|f'(a)||f'(b)|^{-1})^{sq}]^{1/q}. \end{aligned}$$

Hence, the proof of the Theorem 3.4 is complete. $\square$

If putting $s = 1$ in Theorem 3.4, we have

Corollary 3.5. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the numbers $a, b \in I^\circ$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'(x)|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q > 1$ with $\frac{1}{q} + \frac{1}{p} = 1$, then*

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{2(p+1)^{1/p}} |f'(b)| [N(|f'(a)||f'(b)|^{-1})^q]^{1/q}, \end{aligned}$$

where $N(\alpha)$ is defined in (4).

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