

A geometric inequality in acute triangles and its applications

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Abstract

In this paper, we obtain an improvement of a known linear geometric inequality in acute triangles. We also give several applications of the improved result.

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1 Introduction

Given a triangle ABC , let a, b, c be the side lengths of ABC , m_a, m_b, m_c the corresponding medians, h_a, h_b, h_c the altitudes, w_a, w_b, w_c the angle-bisectors, and r_a, r_b, r_c the radii of excircles. And let s, S, R and r be its semiperimeter, area, radius of circumcircle and the radius of incircle, respectively. In addition, denote cyclic sums and products by $\sum$ and $\prod$, respectively.

In a Chinese paper [1], the author gave the following linear inequality for any acute triangle ABC :

$$4m_a \geq r_b + r_c + 2h_a, \quad (1)$$

with equality if and only if $b = c$.

It is easy to prove the following identity:

$$r_b + r_c = 2h_a + \frac{(r_b - r_c)^2}{r_b + r_c}. \quad (2)$$

Thus inequality (1) is equivalent to

$$m_a \geq h_a + \frac{(r_b - r_c)^2}{4(r_b + r_c)}. \quad (3)$$

The author has found that inequality (1) or (3) could be used to prove many cyclic triangle inequalities involving medians of an acute triangle. For example, in [1] we used it to prove the following inequality:

$$\sum m_a h_a \geq \sum w_b w_c. \quad (4)$$

For it is natural to assume that refining inequality (2) may make it more applicable, the author first finds the following improvement:

$$4m_a \geq r_b + r_c + 2h_a + \frac{3}{4} \cdot \frac{(b - c)^2}{b + c}. \quad (5)$$

Then we further obtain the following stronger result:

$$4m_a \geq r_b + r_c + 2h_a + \frac{3}{4} \cdot \frac{(b - c)^2}{r_b + r_c}. \quad (6)$$

By identity (2), we see that the above inequality is equivalent to the following inequality:

Theorem 1.1 *In the acute triangle ABC the following inequality holds:*

$$m_a \geq h_a + \frac{3(b - c)^2 + 4(r_b - r_c)^2}{16(r_b + r_c)}, \quad (7)$$

with equality if and only if $b = c$.

For the equivalent form (6) of (7), we wish point out that if replacing the constant $3/4$ by 1 in (6), then it holds reversely for any triangle ABC , i.e., for any triangle ABC holds:

$$4m_a \leq r_b + r_c + 2h_a + \frac{(b - c)^2}{r_b + r_c}, \quad (8)$$

with equality if and only if $b = c$ or $A = \pi/2$. However, inequality (8) is equivalent to

$$m_a \leq h_a + R \left(\frac{b - c}{a} \right)^2, \quad (9)$$

which is the main result in the recent paper [2]. In fact, we can easily prove the following consecutive identity:

$$\frac{1}{4}(r_b + r_c - 2h_a) + \frac{(b - c)^2}{4(r_b + r_c)} = \frac{(b - c)^2 + (r_b - r_c)^2}{4(r_b + r_c)} = R \left(\frac{b - c}{a} \right)^2, \quad (10)$$

which shows that inequality (8) are equivalent to (9) and

$$m_a \leq h_a + \frac{(r_b - r_c)^2 + (b - c)^2}{4(r_b + r_c)}, \quad (11)$$

which has the similar form with inequality (7).

The main purpose of this paper is to prove inequality (7) and give its several applications.

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 as follows:

Proof. By the known formula:

$$r_a = \frac{S}{s - a}, \quad (12)$$

we get

$$r_b + r_c = \frac{aS}{(s - b)(s - c)} \quad (13)$$

and

$$r_b - r_c = \frac{(b - c)S}{(s - b)(s - c)}. \quad (14)$$

Using (13), (14), Heron's formula

$$S = \sqrt{s(s - a)(s - b)(s - c)} \quad (15)$$

and $s = (a + b + c)/2$, we have

$$\begin{aligned} & \frac{4(r_b - r_c)^2 + 3(b - c)^2}{16(r_b + r_c)} \\ &= \frac{[4S^2 + 3(s - b)^2(s - c)^2](b - c)^2}{16a(s - b)(s - c)S} \\ &= \frac{[4s(s - a) + 3(s - b)(s - c)](b - c)^2}{16aS} \\ &= \frac{[4(a + b + c)(b + c - a) + 3(c + a - b)(a + b - c)](b - c)^2}{64aS} \\ &= \frac{(b^2 + c^2 - a^2 + 14bc)(b - c)^2}{64aS}. \end{aligned}$$

Therefore, inequality (7) is equivalent to

$$m_a \geq h_a + \frac{(b^2 + c^2 - a^2 + 14bc)(b - c)^2}{64aS}. \quad (16)$$

Note that $ah_a = 2S$, to prove the above inequality we need to prove

$$64am_aS \geq 128S^2 + (b^2 + c^2 + 14bc - a^2)(b - c)^2. \quad (17)$$

It is obvious that the value of the right hand side of (17) is positive for the acute triangle ABC . Thus, we only need to prove the following inequality:

$$D_0 \equiv (64am_aS)^2 - [128S^2 + (b^2 + c^2 + 14bc - a^2)(b - c)^2]^2 \geq 0.$$

Using the known median formula:

$$4m_a^2 = 2b^2 + 2c^2 - a^2 \quad (18)$$

and the equivalent form of Heron's formula, i.e.,

$$S = \frac{1}{4} \sqrt{2 \sum b^2 c^2 - \sum a^4}, \quad (19)$$

we easily obtain

$$D_0 = (b - c)^2 D_1, \quad (20)$$

where

$$D_1 = -16a^6 - (17b^2 - 98bc + 17c^2)a^4 + 2(41b^2 - 2bc + 41c^2)(b - c)^2 a^2 - (b - c)^2 (7b^2 + 2bc + 7c^2)^2.$$

Therefore, to prove inequality $D_0 \geq 0$ it remains to show that $D_1 > 0$. Through analysis, the author finds that D_1 can be rewritten as

$$\begin{aligned} D_1 = & 16(a + b + c)(b + c - a)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2) \\ & + 3(c + a - b)(a + b - c)(11b^2 + 11c^2 - 11a^2 \\ & + 10bc)(b - c)^2, \end{aligned} \quad (21)$$

which can be easily verified by expanding. From (21) we see that $D_1 > 0$ strictly holds for the acute triangle ABC . Hence, inequality (7) is proved. Also, from identity (20) we deduce that equality in (9) occurs if and only if $b = c$. This completes the proof of Theorem 1.1.

Remark 2.1 *Inequality (7) is better than (5). This conclusion can be proved as follows:*

Since (7) is equivalent to (6), it suffice to prove the following simple linear inequality

$$r_b + r_c \leq b + c \quad (22)$$

for the acute triangle ABC . By the previous identity (13), we need to prove

$$(b + c)(s - b)(s - c) \geq aS, \quad (23)$$

Squaring both sides and using Heron's formula, one sees that the claimed inequality is equivalent to

$$(s-b)(s-c)(b+c)^2 \geq (s-a)a^2, \quad (24)$$

which follows from the following identity:

$$(s-b)(s-c)(b+c)^2 - s(s-a)a^2 = \frac{1}{4}(c^2 + a^2 - b^2)(a^2 + b^2 - c^2). \quad (25)$$

This completes the proof.

3 Applications of Theorem 1.1

In this section, we shall apply Theorem 1.1 to establish several new triangle inequalities. For simplicity, we shall omit the details of deducing some identities in a triangle.

In [1], the author proved the following inequality:

$$\sum \frac{r_b + r_c}{m_a + r_a} \geq 3, \quad (26)$$

which holds for any triangle ABC . For the acute triangle ABC , we shall apply Theorem 1.1 to establish the following reverse inequality:

Theorem 3.1 *In the acute triangle ABC the following inequality holds:*

$$\sum \frac{h_a}{r_a} \geq \sum \frac{r_b + r_c}{m_a + r_a}, \quad (27)$$

with equality if and only if the triangle ABC is equilateral.

Proof. We set

$$u_a = h_a + \frac{4(r_b - r_c)^2 + 3(b - c)^2}{16(r_b + r_c)}, \quad (28)$$

$$u_b = h_b + \frac{4(r_c - r_a)^2 + 3(c - a)^2}{16(r_c + r_a)}, \quad (29)$$

$$u_c = h_c + \frac{4(r_a - r_b)^2 + 3(a - b)^2}{16(r_a + r_b)}. \quad (30)$$

By Theorem 1.1, we have $m_a \geq u_a, m_b \geq u_b, m_c \geq u_c$ in the acute triangle ABC . Thus, to prove (27) we only need to show that

$$\sum \frac{h_a}{r_a} \geq \sum \frac{r_b + r_c}{u_a + r_a}. \quad (31)$$

Since $h_a/r_a = 2(s - a)/a$, so that

$$\sum \frac{h_a}{r_a} = 2 \sum \frac{s - a}{a},$$

then using the follow known basic identities (see, e.g., [3]):

$$\sum a = 2s, \quad (32)$$

$$\sum bc = s^2 + 4Rr + r^2, \quad (33)$$

$$abc = 4Rrs, \quad (34)$$

we easily get

$$\sum \frac{h_a}{r_a} = \frac{s^2 - 8Rr + r^2}{2Rr}. \quad (35)$$

Next, we compute the right hand side of (31) in term of R, r and s . By applying (32)-(34) and some other known identities (cf. [4] and [5]), we can obtain the following two identities:

$$\prod (u_a + r_a) = \frac{M_1}{4096Rs^2}, \quad (36)$$

$$\sum (r_b + r_c)(u_b + r_b)(u_c + r_c) = \frac{N_1}{128Rs^2}, \quad (37)$$

where

$$\begin{aligned} M_1 &= 900s^6 + (8464R^2 + 1312Rr + 229r^2)s^4 + (19456R^4 - 7040R^3r \\ &\quad - 9204R^2r^2 - 3524Rr^3 - 674r^4)s^2 - 3(16R + r)^2(4R + r)^3r, \\ N_1 &= -60s^6 + (818R^2 + 848Rr + 213r^2)s^4 + 2(4R + r)(608R^3 + 162R^2r \\ &\quad - 120Rr^2 - 61r^3)s^2 - 3(16R + r)(4R + r)^4r. \end{aligned}$$

Now, it follows from (36) and (37) that

$$\sum \frac{r_b + r_c}{u_a + r_a} = \frac{32N_1}{M_1}. \quad (38)$$

Noting that $M_1 > 0$, one sees that inequality (31) is equivalent to

$$(s^2 - 8Rr + r^2)M_1 - 64RrN_1 \geq 0. \quad (39)$$

Substituting the expressions of M_1 and N_1 into (39) and arranging gives

$$\begin{aligned} P_0 &\equiv 900s^8 + (8464R^2 - 2048Rr + 1129r^2)s^6 + (19456R^4 - 127104R^3r \\ &\quad - 65508R^2r^2 - 17676Rr^3 - 445r^4)s^4 - (516096R^5 + 128000R^4r \\ &\quad - 93280R^3r^2 - 63516R^2r^3 - 9544Rr^4 + 677r^5)rs^2 \\ &\quad + 3(16R + r)(384R^2 + 56Rr - r^2)(4R + r)^3r^2 \geq 0, \end{aligned} \quad (40)$$

which needs to be proved.

Now, recall that for the acute triangle ABC we have Walker's inequality (see [6], [3] (p.247) and [7]):

$$w_0 \equiv s^2 - 2R^2 - 8Rr - 3r^2 \geq 0. \quad (41)$$

Then we can rewrite P_0 as

$$\begin{aligned} P_0 = & 900w_0^4 + (15664R^2 + 26752Rr + 11929r^2)w_0^3 + (91840R^4 + 236544R^3r \\ & + 378690R^2r^2 + 250188Rr^3 + 58316r^4)w_0^2 + P_1, \end{aligned} \quad (42)$$

where

$$\begin{aligned} P_1 = & (208192R^6 + 420352R^5r + 951772R^4r^2 + 1952080R^3r^3 \\ & + 1844900R^2r^4 + 781248Rr^5 + 124336r^6)s^2 \\ & - (256448R^8 + 2397696R^7r + 7110656R^6r^2 \\ & + 14184976R^5r^3 + 20535108R^4r^4 + 18420752R^3r^5 \\ & + 9432284R^2r^6 + 2529408Rr^7 + 275664r^8). \end{aligned}$$

Since we have $w_0 \geq 0$, it remains to prove $P_1 \geq 0$. For the sake of convenience, we set $P_1 = f(s)$. We shall consider the following two cases to finish the proof of inequality $f(s) \geq 0$.

Case 1. R and r satisfy $R^2 - 2Rr - r^2 > 0$.

In this case, by Ciamberlini's acute triangle inequality (see [8] and [9]):

$$s \geq 2R + r, \quad (43)$$

to prove $f(s) \geq 0$ we only need to show $f(2R + r) > 0$. But it is easy to check the following identity:

$$\begin{aligned} f(2R + r) &= (R^2 - 2Rr - r^2)(576320R^6 + 1269120R^5r + 1700592R^4r^2 \\ &\quad + 2521088R^3r^3 + 2347352R^2r^4 + 1251712Rr^5 \\ &\quad + 885728r^6) + 9792(181R + 75r)r^7. \end{aligned}$$

Thus, one sees that $f(2R + r) > 0$ holds under Case 1.

Case 2. R and r satisfy $R^2 - 2Rr - r^2 \leq 0$.

In this case, by the author's result for the acute triangle ABC (cf. [7]):

$$s^2 \geq 16Rr - 3r^2 - \frac{4r^3}{R}, \quad (44)$$

to prove $f(s) \geq 0$ we only need to show that

$$f\left(\sqrt{16Rr - 3r^2 - \frac{4r^3}{R}}\right) \geq 0.$$

A simple calculations gives equivalent inequality:

$$\frac{8(R-2r)}{R}P_2 \geq 0,$$

where

$$P_2 = -32056R^8 + 52560R^7r - 21080R^6r^2 - 173466R^5r^3 + 423249R^4r^4 \\ + 1025788R^3r^5 + 767159R^2r^6 + 251396Rr^7 + 31084r^8.$$

In view of Euler's inequality (valid for any triangle ABC):

$$R \geq 2r, \tag{45}$$

it remains to show that $P_2 > 0$. We can rewrite P_2 as follows:

$$P_2 = -(R^2 - 2Rr - r^2)(32056R^6 + 11552R^5r + 76240R^4r^2 + 337498R^3r^3 \\ + 327987R^2r^4 - 32316Rr^5 - 503804r^6) + 1224(1055R + 437r)r^7.$$

Thus, by the hypothesis $R^2 - 2Rr - r^2 \leq 0$ and Euler's inequality, one sees that $P_2 > 0$ holds. Hence, $f(s) \geq 0$ is proved under Case 2.

Combining the arguments of the above two cases, we conclude that inequality $f(s) \geq 0$ (i.e. $P_1 \geq 0$) holds for all acute triangles. It is easy to deduce that equality in (27) holds if and only if $\triangle ABC$ is equilateral. This completes the proof of Theorem 3.1.

Inequality (46) below was proposed by the author in [10] as an open problem, which was unproved before. Here, we shall use Theorem 1.1 to finish its proof.

Theorem 3.2 *In the acute triangle ABC the following inequality holds:*

$$\sum \frac{1}{m_a + r_a} \leq \frac{3\sqrt{3}}{2s}, \tag{46}$$

with equality if and only if triangle ABC is equilateral.

Proof. By Theorem 1.1, we only need to prove

$$\sum \frac{1}{u_a + r_a} \leq \frac{3\sqrt{3}}{2s}, \tag{47}$$

where u_a, u_b, u_c are given by (28), (29) and (30), respectively. Next, we shall prove that inequality (47) holds for any triangle ABC .

By calculation, we obtain the following identities:

$$\prod (u_a + r_a) = \frac{M_2}{4096Rs^2}, \tag{48}$$

$$\sum (u_b + r_b)(u_c + r_c) = \frac{N_2}{256Rs^2}, \tag{49}$$

where

$$\begin{aligned} M_2 = & 900s^6 + (8464R^2 + 1312Rr + 229r^2)s^4 + (19456R^4 - 7040R^3r \\ & - 9204R^2r^2 - 3524Rr^3 - 674r^4)s^2 - 3(16R + r)^2(4R + r)^3r, \\ N_2 = & (593R + 232r)s^4 + (2688R^3 + 488R^2r - 467Rr^2 - 166r^3)s^2 \\ & - 6(16R + r)(4R + r)^3r, \end{aligned}$$

It follows from (48) and (49) that

$$\sum \frac{1}{u_a + r_a} = \frac{16N_2}{M_2}. \quad (50)$$

Thus, to prove inequality (47) we only need to show that

$$\frac{27}{4s^2} \geq \frac{256N_2^2}{M_2^2},$$

i.e.,

$$Q_0 \equiv 27M_2^2 - 1024s^2N_2^2 \geq 0. \quad (51)$$

Using the expressions of M_2 and N_2 , we get the following equivalent inequality:

$$\begin{aligned} Q_0 \equiv & 21870000s^{12} + (51261824R^2 - 217992448Rr - 43986376r^2)s^{10} \\ & + (-384656640R^4 - 1612310528R^3r + 39116032R^2r^2 \\ & + 268447328Rr^3 + 47532083r^4)s^8 + (1493729280R^6 \\ & + 547172352R^5r + 4753717376R^4r^2 + 2470287488R^3r^3 \\ & + 58955880R^2r^4 - 169399960Rr^5 - 33847012r^6)s^6 \\ & + (10220470272R^8 + 3961257984R^7r + 2150375424R^6r^2 \\ & - 3224208384R^5r^3 - 4048552656R^4r^4 - 1245597216R^3r^5 \\ & - 10191456R^2r^6 + 63561528Rr^7 + 10188546r^8)s^4 \\ & - 36(87552R^4 + 33856R^3r + 7734R^2r^2 - 3570Rr^3 \\ & - 2009r^4)(16R + r)^2(4R + r)^3rs^2 \\ & + 243(16R + r)^4(4R + r)^6r^2 \geq 0, \end{aligned} \quad (52)$$

which needs to be proved.

Now, we first note that

$$87552R^4 + 33856R^3r + 7734R^2r^2 - 3570Rr^3 - 2009r^4 > 0,$$

which follows from Euler's inequality $R \geq 2r$. Dividing both sides of (52) by s^4 , and noting the following two known inequalities (see [10]):

$$3r(4R + r) \leq s^2, \quad (53)$$

and

$$(4R + r)^2 \geq 3s^2, \quad (54)$$

we only need to prove that

$$\begin{aligned} & 21870000s^8 + (51261824R^2 - 217992448Rr - 43986376r^2)s^6 \\ & + (-384656640R^4 - 1612310528R^3r + 39116032R^2r^2 \\ & + 268447328Rr^3 + 47532083r^4)s^4 + (1493729280R^6 \\ & + 547172352R^5r + 4753717376R^4r^2 + 2470287488R^3r^3 \\ & + 58955880R^2r^4 - 169399960Rr^5 - 33847012r^6)s^2 \\ & + (10220470272R^8 + 3961257984R^7r + 2150375424R^6r^2 \\ & - 3224208384R^5r^3 - 4048552656R^4r^4 - 1245597216R^3r^5 \\ & - 10191456R^2r^6 + 63561528Rr^7 + 10188546r^8) \\ & - 12(87552R^4 + 33856R^3r + 7734R^2r^2 - 3570Rr^3 \\ & - 2009r^4)(16R + r)^2(4R + r)^2 \\ & + 243(16R + r)^4 \cdot 9(4R + r)^2r^2 \geq 0. \end{aligned}$$

After simplifying and rearranging, the above inequality becomes

$$\begin{aligned} Q_1 \equiv & 21870000s^8 + (51261824R^2 - 217992448Rr - 43986376r^2)s^6 \\ & - (384656640R^4 + 1612310528R^3r - 39116032R^2r^2 \\ & - 268447328Rr^3 - 47532083r^4)s^4 + (1493729280R^6 \\ & + 547172352R^5r + 4753717376R^4r^2 + 2470287488R^3r^3 \\ & + 58955880R^2r^4 - 169399960Rr^5 - 33847012r^6)s^2 \\ & + 5917114368R^8 - 392429568R^7r + 2468683776R^6r^2 \\ & - 1822934016R^5r^3 - 3422710608R^4r^4 - 1100434656R^3r^5 \\ & + 8672328R^2r^6 + 64726152Rr^7 + 10214841r^8 \geq 0. \end{aligned} \quad (55)$$

Now, recall that for any triangle ABC we have the following fundamental inequality (see [3] and [9]):

$$t_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \geq 0. \quad (56)$$

Also, we have the following two Gerretsen's inequalities (see [3] and [9]):

$$g_1 \equiv s^2 - 16Rr + 5r^2 \geq 0, \quad (57)$$

and

$$g_2 \equiv 4R^2 + 4Rr + 3r^2 - s^2 \geq 0. \quad (58)$$

Based on these three important inequalities, after analysing we obtains the following identity:

$$Q_1 = g_0g_2t_0 + a_0g_1^2 + b_0g_1 + c_0, \quad (59)$$

where

$$\begin{aligned}
g_0 &= 21870000s^2 + 226221824R^2 + 306887552Rr - 22116376r^2, \\
a_0 &= 1075197952R^4 + 2772913664R^3r + 4566868096R^2r^2 \\
&\quad - 1355138144Rr^3 + 134765707r^4, \\
b_0 &= -2125819904R^6 - 553465856R^5r + 13923308032R^4r^2 \\
&\quad + 78880036480R^3r^3 - 100277958264R^2r^4 + 21519169928Rr^5 \\
&\quad - 1426475962r^6, \\
c_0 &= 16(R - 2r)(369819648R^7 + 2208841728R^6r - 1275926272R^5r^2 \\
&\quad - 7837180224R^4r^3 + 22057895883R^3r^4 - 14134956480R^2r^5 \\
&\quad + 2282006660Rr^6 - 115846964r^7).
\end{aligned}$$

Note that (56), (57), (58) and $g_0 > 0$ (which follows from Euler's inequality). It remains to prove that

$$Q_2 \equiv a_0g_1^2 + b_0g_1 + c_0 \geq 0. \quad (60)$$

We consider the following two cases to complete the proof of inequality (60).

Case 1. R and r satisfy $2r \leq R \leq 3r$.

By Euler's inequality $R \geq 2r$, it can be seen that $a_0 > 0$ and $c_0 > 0$. Since $g_1 \geq 0$ again, thus to prove inequality (60) under Case 1 we only need to show $b_0 > 0$. Since

$$\begin{aligned}
\frac{b_0}{553465856} &= -(3.84 \dots)R^6 - R^5r + (25.15 \dots)R^4r^2 + (142.52 \dots)R^3r^3 \\
&\quad - (181.18 \dots)R^2r^4 + (38.88 \dots)Rr^5 - (2.57 \dots)r^6,
\end{aligned}$$

it is enough to show that

$$-4R^6 - R^5r + 25R^4r^2 + 142R^3r^3 - 182R^2r^4 + 38Rr^5 - 3r^6 > 0.$$

By Euler's inequality we have $100R^3r^3 - 182R^2r^4 + 38Rr^5 - 3r^6 > 0$. Thus, it remains to prove

$$-4R^6 - R^5r + 25R^4r^2 + 42R^3r^3 \geq 0.$$

Under the hypothesis, we may assume that $r = R/3 + t$ ($t \geq 0$) and substitute this into the above inequality, then it becomes

$$\frac{89}{3}R^5t + 67R^4t^2 + 42R^3t^3 \geq 0,$$

which is clearly true. Hence, inequality (60) is proved under Case 1.

Case 2. R and r satisfy $R > 3r$.

Q_2 is a quadratic function in g_1 (with $a_0 > 0$ and $c_0 > 0$). We only need to prove its discriminant F_0 is negative under the hypothesis. With the help of Maple, it is easy to obtain

$$\begin{aligned}
-F_0 &= 4a_0c_0 - b_0^2 \\
&= 20929166736650928128R^{12} + 164377192205601734656R^{11}r \\
&\quad + 35922984254434181120R^{10}r^2 - 625996886933597323264R^9r^3 \\
&\quad - 662758922643395248128R^8r^4 - 566551317731928768512R^7r^5 \\
&\quad - 209448620498416926720R^6r^6 - 24613080705247551488R^5r^7 \\
&\quad + 81775339985542882816R^4r^8 + 61982613365085811968R^3r^9 \\
&\quad - 12054142910647444384R^2r^{10} + 934791847340434592Rr^{11} \\
&\quad - 36472325234091300r^{12},
\end{aligned}$$

and then

$$\begin{aligned}
&-F_0/36472325234091300 \\
&= (573.83 \dots)R^{12} + (4506.90 \dots)R^{11}r + (984.93 \dots)R^{10}r^2 \\
&\quad - (17163.61 \dots)R^9r^3 - (18171.55 \dots)R^8r^4 - (15533.73 \dots)R^7r^5 \\
&\quad - (5742.67 \dots)R^6r^6 - (674.84 \dots)R^5r^7 + (2242.12 \dots)R^4r^8 \\
&\quad + (1699.44 \dots)R^3r^9 - (330.50 \dots)R^2r^{10} + (25.63 \dots)Rr^{11} - r^{12}.
\end{aligned}$$

Thus, by Euler's inequality, to prove $F_0 < 0$ we only need to show that

$$\begin{aligned}
&573R^{12} + 4506R^{11}r + 984R^{10}r^2 - 17164R^9r^3 - 18172R^8r^4 \\
&\quad - 15534R^7r^5 - 5743R^6r^6 - 675R^5r^7 + 2242R^4r^8 > 0.
\end{aligned}$$

Dividing both sides by 573, the above inequality becomes

$$\begin{aligned}
&(1.00 \dots)R^{12} + (7.86 \dots)R^{11}r + (1.71 \dots)R^{10}r^2 \\
&\quad - (29.95 \dots)R^9r^3 - (31.71 \dots)R^8r^4 - (27.10 \dots)R^7r^5 \\
&\quad - (10.02 \dots)R^6r^6 - (1.17 \dots)R^5r^7 + (3.91 \dots)R^4r^8 > 0,
\end{aligned}$$

Thus, it is enough to show that

$$R^{12} + 7R^{11}r + R^{10}r^2 - 30R^9r^3 - 32R^8r^4 - 28R^7r^5 - 11R^6r^6 - 2R^5r^7 > 0.$$

Under the hypothesis $R > 3r$, we may assume that $R = 3r + t$ ($t > 0$) and substitute it into the above inequality. Expanding gives

$$\begin{aligned}
&t^{12} + 43t^{11}r + 826t^{10}r^2 + 9405t^9r^3 + 70843t^8r^4 + 372290t^7r^5 \\
&\quad + 1399765t^6r^6 + 3790870t^5r^7 + 7324800t^4r^8 + 9812619t^3r^9 \\
&\quad + 8598501t^2r^{10} + 4378293tr^{11} + 960336r^{12} > 0,
\end{aligned}$$

which is true. Thus, the claimed inequality $F_0 < 0$ and inequality (60) are proved under Case 2.

Combining the arguments of the above two cases, we conclude that inequality (60) holds for any triangle ABC . We therefore proved that (47) holds for any triangle ABC . From the above proof, one can easily conclude that equality in (46) holds if and only if triangle ABC is equilateral. This completes the proof of Theorem 3.2.

Finally, we apply Theorem 1.1 to prove a linear inequality in acute triangles.

Theorem 3.3 *In the acute triangle ABC , the following inequality holds:*

$$2m_a + r_a \geq m_b + m_c + w_a, \quad (61)$$

with equality if and only if triangle ABC is equilateral.

Proof. Note that

$$m_b \leq \frac{1}{4}(r_c + r_a) + \frac{m_b^2}{r_c + r_a}, \quad (62)$$

$$m_c \leq \frac{1}{4}(r_a + r_b) + \frac{m_c^2}{r_a + r_b}, \quad (63)$$

and

$$h_a \leq \frac{1}{2} \left(h_a + \frac{w_a^2}{h_a} \right), \quad (64)$$

It suffice to prove that

$$2m_a + \frac{1}{2}r_a \geq \frac{1}{2}h_a + \frac{1}{4}(r_b + r_c) + \frac{w_a^2}{2h_a} + \frac{m_b^2}{r_c + r_a} + \frac{m_c^2}{r_a + r_b}.$$

By Theorem 1.1, we only need to prove

$$\begin{aligned} W_0 &\equiv \frac{3}{2}h_a + \frac{1}{2}r_a + \frac{4(r_b - r_c)^2 + 3(b - c)^2}{8(r_b + r_c)} \\ &\quad - \frac{1}{4}(r_b + r_c) - \frac{w_a^2}{2h_a} - \frac{m_b^2}{r_c + r_a} - \frac{m_c^2}{r_a + r_b} \geq 0. \end{aligned} \quad (65)$$

Using $h_a = 2S/a$ and $r_a = S/(s - a)$, the following known formulas:

$$w_a^2 = \frac{4s(s - a)bc}{(b + c)^2}, \quad (66)$$

$$m_a^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2, \quad (67)$$

and Hern's formula (15), we easily obtain

$$W_0 = \frac{W_1}{64abcS}, \quad (68)$$

where

$$\begin{aligned} W_1 \equiv & 4(b+c)^3a^5 - (8b^4 + 18b^3c + 12b^2c^2 + 18bc^3 \\ & + 8c^4)a^4 + 4(2b^2 - 3bc + 2c^2)(b+c)^3a^3 - (8b^2 + 9bc \\ & + 8c^2)(b-c)^2(b+c)^2a^2 + 4(b^2 + bc + c^2)(b-c)^2(b+c)^3a \\ & - (5b^2 - 2bc + 5c^2)(b-c)^2(b+c)^2bc. \end{aligned}$$

Therefore, we have to prove $W_1 \geq 0$. Next, we shall use a very effective method (cf. [11]) to prove this inequality.

Putting $s-a = x, s-b = y, s-c = z$, then we have $a = y+z, b = z+x, c = x+y$ ($x, y, z > 0$). Submitting them into W_1 , we easily get

$$W_1 = 4F(x, y, z), \quad (69)$$

where

$$\begin{aligned} F(x, y, z) \equiv & -8(y-z)^2x^6 + 8(y+z)^3x^5 + (8y^4 + 20y^3z + 8y^2z^2 \\ & + 20yz^3 + 8z^4)x^4 - 16(y+z)^3yzx^3 - (13y^4 + 20y^3z \\ & + 62y^2z^2 + 20yz^3 + 13z^4)yzx^2 - (y+z)(y^4 - 20y^3z \\ & + 6y^2z^2 - 20yz^3 + z^4)yzx + (7y^2 + 2yz \\ & + 7z^2)(y+z)^2y^2z^2. \end{aligned}$$

Since the triangle ABC is an acute triangle, thus the positive numbers x, y, z satisfy the following three relations:

$$(z+x)^2 + (x+y)^2 - (y+z)^2 > 0, \quad (70)$$

$$(x+y)^2 + (y+z)^2 - (z+x)^2 > 0, \quad (71)$$

$$(y+z)^2 + (z+x)^2 - (x+y)^2 > 0. \quad (72)$$

Under these three relations, it remains to prove the following inequality:

$$F(x, y, z) \geq 0. \quad (73)$$

Note that y and z are symmetric with respect to $F(x, y, z)$. To prove $F(x, y, z) \geq 0$ we may assume that $y \geq z$ and consider the following three cases to complete the proof.

Case 1. The positive numbers x, y and z satisfy $x \geq y \geq z$.

In this case, we may set $y = z + m, x = z + m + n$ ($m \geq 0, n \geq 0$). Substituting them into (72) and simplifying gives

$$2z^2 - m^2 - mn \geq 0. \quad (74)$$

Also, with the help of Maple we easily obtain

$$\begin{aligned}
& F(z + m + n, z + m, z) \\
&= (160m^2 + 512mn + 512n^2)z^6 + (656m^3 + 2272m^2n + 2688mn^2 \\
&\quad + 768n^3)z^5 + (1084m^4 + 4072m^3n + 5496m^2n^2 + 2688mn^3 \\
&\quad + 384n^4)z^4 + (918m^5 + 3716m^4n + 5560m^3n^2 + 3568m^2n^3 \\
&\quad + 928mn^4 + 64n^5)z^3 + (416m^5 + 1769m^4n + 2819m^3n^2 \\
&\quad + 2080m^2n^3 + 716mn^4 + 96n^5)mz^2 + (m + n)(94m^3 + 295m^2n \\
&\quad + 292mn^2 + 92n^3)m^3z + 8(m^2 - mn - n^2)(m + n)^4m^2. \tag{75}
\end{aligned}$$

And it is easy to verify the following identity:

$$2F(z + m + n, z + m, z) = (2z^2 - m^2 - mn)Z_1 + Z_2, \tag{76}$$

where

$$\begin{aligned}
Z_1 &= m(416m^5 + 1769m^4n + 2819m^3n^2 + 2080m^2n^3 + 716mn^4 + 96n^5), \\
Z_2 &= (320m^2 + 1024mn + 1024n^2)z^6 + (1312m^3 + 4544m^2n + 5376mn^2 \\
&\quad + 1536n^3)z^5 + (2168m^4 + 8144m^3n + 10992m^2n^2 + 5376mn^3 \\
&\quad + 768n^4)z^4 + (1836m^5 + 7432m^4n + 11120m^3n^2 + 7136m^2n^3 \\
&\quad + 1856mn^4 + 128n^5)z^3 + 2(m + n)(94m^3 + 295m^2n + 292mn^2 \\
&\quad + 92n^3)m^3z + (m + n)(432m^5 + 1801m^4n + 2803m^3n^2 \\
&\quad + 2000m^2n^3 + 652mn^4 + 80n^5)m^2.
\end{aligned}$$

Note that $m \geq 0, n \geq 0$ and $z > 0$. Thus, from (74) and (76) we deduce that $F(z + m + n, z + m, z) \geq 0$ holds. This completes the proof of (73) under Case 1.

Case 2. The positive numbers x, y and z satisfy $y \geq x \geq z$.

In this case, we may set $x = z + m, y = z + m + n (m \geq 0, n \geq 0)$. Substituting them into (72) and simplifying, we get the above inequality (74) again. Also, we can obtain

$$\begin{aligned}
& F(z + m, z + m + n, z) \\
&= (160m^2 - 192mn + 160n^2)z^6 + (656m^3 - 304m^2n + 112mn^2 \\
&\quad + 304n^3)z^5 + (1084m^4 + 264m^3n - 216m^2n^2 + 424mn^3 \\
&\quad + 204n^4)z^4 + (918m^5 + 874m^4n - 124m^3n^2 - 4m^2n^3 \\
&\quad + 198mn^4 + 58n^5)z^3 + (m + n)(416m^5 + 311m^4n - 97m^3n^2 \\
&\quad - 77m^2n^3 + 17mn^4 + 6n^5)z^2 + (94m^3 - 13m^2n - 16mn^2 \\
&\quad - n^3)(m + n)^3mz + 8(m^2 + 3mn + n^2)(m + n)^2m^4. \tag{77}
\end{aligned}$$

And it is easy to verify the following identity:

$$2F(z + m, z + m + n, z) = (2z^2 - m^2 - mn)Z_3 + m(m + n)Z_4, \quad (78)$$

where

$$\begin{aligned} Z_3 = & (160m^2 - 192mn + 160n^2)z^4 + (656m^3 - 304m^2n + 112mn^2 \\ & + 304n^3)z^3 + (1164m^4 + 248m^3n - 232m^2n^2 + 504mn^3 \\ & + 204n^4)z^2 + (1246m^5 + 1050m^4n - 220m^3n^2 + 204m^2n^3 \\ & + 350mn^4 + 58n^5)z + (m + n)(6n^5 + 119mn^4 + 175m^2n^3 \\ & - 213m^3n^2 + 435m^4n + 998m^5), \\ Z_4 = & (1434m^5 + 1400m^4n - 116m^3n^2 + 112m^2n^3 + 314mn^4 \\ & + 56n^5)z + (m + n)(6n^5 + 119mn^4 + 175m^2n^3 - 197m^3n^2 \\ & + 483m^4n + 1014m^5). \end{aligned}$$

It is easy to show that

$$\begin{aligned} 160m^2 - 192mn + 160n^2 &\geq 0, \\ 656m^2 - 304mn + 112n^2 &\geq 0, \\ 248m^2 - 232mn + 504n^2 &\geq 0, \\ 1050m^2 - 220mn + 204n^2 &\geq 0, \\ 1400m^2 - 116mn + 112n^2 &\geq 0, \\ 175n^2 - 197mn + 483m^2 &\geq 0. \end{aligned}$$

So, we have $Z_3 \geq 0$ and $Z_4 \geq 0$. Thus, by inequality (74) (which is still valid under Case 2) and identity (78) we deduce that $F(z + m, z + m + n, z) \geq 0$ holds.

Case 3. The positive numbers x, y and z satisfy $y \geq z \geq x$.

In this case, we may set $z = x + m, y = x + m + n (m \geq 0, n \geq 0)$. Substituting them into $F(x, y, z)$, we easily obtain

$$\begin{aligned} & F(x, x + m + n, x + m) \\ = & (512m^2 + 512mn + 160n^2)x^6 + 16(2m + n)(72m^2 + 72mn \\ & + 19n^2)x^5 + (4224m^4 + 8448m^3n + 6136m^2n^2 \\ & + 1912mn^3 + 204n^4)x^4 + 2(2m + n)(1008m^4 + 2016m^3n \\ & + 1404m^2n^2 + 396mn^3 + 29n^4)x^3 + (2112m^6 + 6336m^5n \\ & + 7564m^4n^2 + 4568m^3n^3 + 1415m^2n^4 + 187mn^5 + 6n^6)x^2 \\ & + (2m + n)(m + n)(288m^4 + 576m^3n + 436m^2n^2 + 148mn^3 \\ & + 13n^4)mx + (16m^2 + 16mn + 7n^2)(2m + n)^2(m + n)^2m^2. \end{aligned} \quad (79)$$

Clearly, one sees that $F(x, x + m + n, x + m) \geq 0$ holds.

Finally, combining the arguments of the above three cases, we conclude that inequality (73) holds under the conditions (70)-(72). And, we finish the proofs of $W_1 \geq 0$ and inequality (61). In addition, it is easily seen that equality in (61) holds if and only if triangle ABC is equilateral. This completes the proof of Theorem 3.3.

Remark 3.4 *Inspired by inequality (61), the author found and proved the following refinement of Euler's inequality (45):*

$$R \geq \frac{1}{9}(2h_a + r_a) \geq 2r. \quad (80)$$

4 Four Open Problems

In this section, we propose several conjectures as open problems.

Considering generalizations of Theorem 3.1, we propose the following conjecture:

Conjecture 4.1 *Let $k > 1$ be a real number, then for the acute triangle ABC the following inequality holds:*

$$\sum \left(\frac{h_a}{r_a} \right)^k \geq \sum \left(\frac{r_b + r_c}{m_a + r_a} \right)^k. \quad (81)$$

For the previous inequality (26), the author propose the following stronger inequality:

Conjecture 4.2 *For any triangle ABC the following inequality holds:*

$$\sum \sqrt{\frac{r_b + r_c}{m_a + r_a}} \geq 3. \quad (82)$$

It seems likely that the following similar inequality holds:

Conjecture 4.3 *For any triangle ABC the following inequality holds:*

$$\sum \sqrt{\frac{r_b + r_c}{m_b + m_c}} \geq 3. \quad (83)$$

Finally, we give the following conjecture:

Conjecture 4.4 *For the acute triangle ABC the following inequality holds:*

$$\sum \frac{1}{(m_a + r_a)^2} \leq \frac{9}{4s^2}. \quad (84)$$

If the above inequality holds, then by the power mean inequality we immediately deduce that inequality (46) of Theorem 3.2 holds. Incidentally, the author has known that inequality (84) can not be proved by using Theorem 1.1.

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