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T -Fuzzy B -subalgebras and normal T -Fuzzy B -subalgebras of B -algebras

R. Rasuli

Department of Mathematics, Payame Noor University (PNU),

P. O. Box 19395-3697, Tehran, Iran.

e-mail:rasulirasul@yahoo.com

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Abstract

In this note, as using t -norms, the fuzzification of B -subalgebras and normal B -subalgebras are considered, and some related properties of them are investigated and characterizations of them are given. Next their family of level subalgebras of them will be classified. Later, the concepts of intersection and direct product of them will be defined and obtained some results. Finally, they will be considered and investigated under B -homomorphisms of B -algebras.

Keywords: *Algebra and orders, theory of fuzzy sets, norms, products and intersections, homomorphisms.*

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1 Introduction

Neggers and Kim [13] introduced the notion of a B -algebra, and investigated several properties. They established the fundamental homomorphism theorem of B -algebras in [12]. Cho and Kim [13, 4] discussed B -algebras related to quasigroups. The concept of fuzzy sets was proposed by Zadeh [33]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations

of fuzzy sets. Jun et al. [5] defined the notion of fuzzy B -algebra. Kim et al. [6] constructed a quotient B -algebra by using a fuzzy normal B -algebra. Triangular norms (t -norms for short) were introduced by Schweizer and Sklar in [28-31], following some ideas of Menger in the context of probabilistic metric spaces [11] (as statistical metric spaces were called after 1964). With the development of t -norms in statistical metric spaces, they also play an important role in decision making, in statistics as well as in the theories of cooperative games. In particular, in fuzzy set theory, t -norms have been widely used for fuzzy operations, fuzzy logic and fuzzy relation equations [32]. In recent years, a systematic study concerning the properties and related matters of t -norms have been made by Klement et al. [7, 8, 9]. The author by using norms, investigated some properties of fuzzy algebraic structures [14-27]. In this study, we introduce the concepts of fuzzy B -algebras and normal fuzzy B -algebra with respect to t -norms and we consider the properties of them. Next we link them with B -algebra by using upper level of fuzzy subsets of B -algebras. Later, we define the intersection and direct product of them and we prove some results about them. Finally, we discuss the image and pre image of them under B -homomorphisms of B -algebras.

2 Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For more details we refer readers to [1, 2, 3, 10, 14].

Definition 2.1 *A B -algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*

- (1) $x * x = 0$,
- (2) $x * 0 = x$,
- (3) $(x * y) * z = x * (z * (0 * y))$

for all $x, y, z \in X$. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$. A non-empty subset N of a B -algebra X is called a B -subalgebra of X if $x * y \in N$ for any $x, y \in N$. A non-empty subset N of a B -algebra X is said to be normal if $(x * a) * (y * b) \in N$ whenever $x * y \in N$ and $a * b \in N$. Note that any normal subset N of a B -algebra X is a B -subalgebra of X , but the converse need not be true. A non-empty subset N of a B -algebra X is called a normal B -subalgebra of X if it is both a B -subalgebra and normal

Example 2.2 *Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by*

$$x * y = \frac{n(x - y)}{n + y}.$$

Then $(X; *, 0)$ is a B -algebra.

Example 2.3 Let Z be the group of integers under usual addition and let $\alpha \notin Z$. We adjoin the special element α to Z . Let $X = Z \cup \alpha$. Define $\alpha + 0 = \alpha$, $\alpha + n = n - 1$ where $n \neq 0$ in Z and $i\alpha + \alpha$ an arbitrary element in X . Define a mapping $\varphi : X \rightarrow X$ by $\varphi(\alpha) = 1, \varphi(n) = n$ where $n \in Z$. If we define a binary operation $*$ on X by $x * y = x + \varphi(y)$, then $(X; *, 0)$ is a non-group derived B-algebra.

Lemma 2.4 If X is a B-algebra, then $x = 0 * (0 * x)$ for all $x \in X$.

Definition 2.5 A mapping $f : (X; *, 0) \rightarrow (Y; \acute{*}, \acute{0})$ of B-algebras is called a B-homomorphism if $f(x * y) = f(x) \acute{*} f(y)$, for all $x, y \in X$. The zero mapping $\theta : (X; *, 0) \rightarrow (Y; \acute{*}, \acute{0})$ of B-algebras with $\theta(0) = \acute{0}$ is a B-homomorphism.

Definition 2.6 Let X be an arbitrary set. A fuzzy subset of X , we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$ -power set of X and is denoted $[0, 1]^X$. For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ .

Definition 2.7 Let φ be a function from set X into set Y such that $\mu : X \rightarrow [0, 1]$ and $\nu : Y \rightarrow [0, 1]$. For all $x \in X, y \in Y$, we define $\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\}$ and $\varphi^{-1}(\nu)(x) = \nu(\varphi(x))$.

Definition 2.8 A t -norm T is a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ having the following four properties:

- (T1) $T(x, 1) = x$ (neutral element),
 - (T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
 - (T3) $T(x, y) = T(y, x)$ (commutativity),
 - (T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
- for all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

- Example 2.9** (1) Standard intersection t -norm $T_m(x, y) = \min\{x, y\}$.
 (2) Bounded sum t -norm $T_b(x, y) = \max\{0, x + y - 1\}$.
 (3) Algebraic product t -norm $T_p(x, y) = xy$.
 (4) Drastic t -norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- (5) Nilpotent minimum t -norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) *Hamacher product t-norm*

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t -norm is the pointwise smallest t -norm and the minimum is the pointwise largest t -norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

We say that T be idempotent if for all $x \in [0, 1]$ we have $T(x, x) = x$.

Definition 2.10 *The function $T_n : \prod_{i=1}^n [0, 1] \rightarrow [0, 1]$ is defined by*

$$T_n(x_1, x_2, \dots, x_n) = T(x_i, T_{n-1}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n))$$

for all $1 \leq i \leq n$, where $n \geq 2$ such that $T_2 = T$ and $T_1 = id$ (identity).

Using the induction on n , we have the following two lemmas.

Lemma 2.11 *For every t -norm T and every $x_i, y_i \in [0, 1]$, where $1 \leq i \leq n$, and $n \geq 2$, we have*

$$T_n(T(x_1, y_1), T(x_2, y_2), \dots, T(x_n, y_n)) = T(T_n(x_1, x_2, \dots, x_n), T_n(y_1, y_2, \dots, y_n)).$$

Lemma 2.12 *For a t -norm T and every $x_1, x_2, \dots, x_n \in [0, 1]$, where $n \geq 2$, we have*

$$T_n(x_1, x_2, \dots, x_n) = T(\dots T(T(T(x_1, x_2), x_3), x_4), x_n) = T(x_1, T(x_2, T(x_3, \dots T(x_{n-1}, x_n) \dots))))$$

Definition 2.13 *Let $\mu, \nu : X \rightarrow [0, 1]$ and T be a t -norm. We define the intersection of μ and ν as*

$$\mu \cap \nu : X \rightarrow [0, 1]$$

by

$$(\mu \cap \nu)(x) = T(\mu(x), \nu(x))$$

for all $x \in X$.

Let $\mu : X \rightarrow [0, 1]$ and $\nu : Y \rightarrow [0, 1]$ and T be a t -norm. The cartesian product of μ and ν is denoted by

$$\mu \times \nu : X \times Y \rightarrow [0, 1]$$

is defined by

$$(\mu \times \nu)(x, y) = T(\mu(x), \nu(y))$$

for all $(x, y) \in X \times Y$.

Lemma 2.14 *Let T be a t -norm. Then*

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z))$$

for all $x, y, w, z \in [0, 1]$.

3 Fuzzy B -algebras under t -norms

Definition 3.1 Let X be a B -algebra. Define $\mu : X \rightarrow [0, 1]$ a fuzzy B -algebra under t -norm T if it satisfies the following inequalities:

$$\mu(x * y) \geq T(\mu(x), \mu(y))$$

for all $x, y \in X$.

Denote by $FBT(X)$, the set of all fuzzy B -algebras of B -algebra X under t -norm T .

Example 3.2 Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is a B -algebra. Define $\mu : X \rightarrow [0, 1]$ as

$$\mu(x) = \begin{cases} 0.8 & \text{if } x = 0, \\ 0.2 & \text{if } x = 1, \\ 0.3 & \text{if } x = 2, \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ for all $a, b \in [0, 1]$ then $\mu \in FBT(X)$.

Proposition 3.3 Let $\mu \in FBT(X)$ and T be idempotent. Then $\mu(0) \geq \mu(x)$ for all $x \in X$.

Proof 3.4 Let $x \in X$ and then $\mu(0) = \mu(x * x) \geq T(\mu(x), \mu(x)) = \mu(x)$.

Proposition 3.5 Let $\mu \in FBT(X)$. Then

- (1) $\mu(0 * x) = \mu(x)$,
 - (2) $\mu(x * (0 * y)) \geq T(\mu(x), \mu(y))$,
- for all $x, y \in X$.

Proof 3.6 Let $x, y \in X$. Then

(1)

$$\mu(0 * x) \geq T(\mu(0), \mu(x)) \geq T(\mu(x), \mu(x)) = \mu(x) = \mu(0 * (0 * x)) \geq T(\mu(0), \mu(0 * x)) = \mu(0 * x)$$

thus $\mu(0 * x) = \mu(x)$.

(2)

$$\mu(x * (0 * y)) \geq T(\mu(x), \mu(0 * y)) = T(\mu(x), \mu(y)).$$

Proposition 3.7 *If a fuzzy set μ in X satisfies (1) and (2) in Proposition 3.4, then $\mu \in FBT(X)$.*

Proof 3.8 *Let $x, y \in X$. Then*

$$\mu(x*y) = \mu(x*(0*(0*y))) \geq T(\mu(x), \mu(0*(0*y))) = T(\mu(x), \mu(0*y)) = T(\mu(x), \mu(y))$$

so $\mu(x * y) \geq T(\mu(x), \mu(y))$ then $\mu \in FBT(X)$.

Proposition 3.9 *Let $\mu : X \rightarrow [0, 1]$ be a fuzzy set and T be idempotent t -norm. If $\mu \in FBT(X)$, then the set*

$$\mu_s = \{x \in X : \mu(x) \geq s\}$$

is either empty or subalgebra of B -algebra X for every $s \in [0, 1]$.

Proof 3.10 *Let $\mu \in FBT(X)$ and $\mu_s = \{x \in X : \mu(x) \geq s\}$ be not empty and $x, y \in \mu_s$. Then $\mu(x) \geq s$ and $\mu(y) \geq s$ such that*

$$\mu(x * y) \geq T(\mu(x), \mu(y)) \geq T(s, s) = s$$

which means that $x * y \in \mu_s$ and so the set μ_s will be subalgebra of B -algebra X for every $s \in [0, 1]$.

Example 3.11 *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set given by the following Cayley table:*

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X, *, 0)$ is a B -algebra. Define

$$\mu : X \rightarrow [0, 1]$$

as

$$\mu(x) = \begin{cases} 0.6 & \text{if } x = 0, \\ 0.5 & \text{if } x = 1, \\ 0.4 & \text{if } x = 2, \\ 0.3 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \\ 0.1 & \text{if } x = 5, \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ for all $a, b \in [0, 1]$ then $\mu \in FBT(X)$.
Let $s = 0.4$ then

$$\mu_{0.3} = \{x \in X : \mu(x) \geq 0.4\} = \{0, 1, 2\}$$

with the following Cayley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

so $\mu_{0.4}$ will be subalgebra of B -algebra X .

Proposition 3.12 Any subalgebra of a B -algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X .

Proof 3.13 Let A be a subalgebra of a given B -algebra X and let $\mu : X \rightarrow [0, 1]$ be a fuzzy subalgebra defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 0, & \text{if } x \notin A, \end{cases}$$

such that $t \in (0, 1)$ is fixed. It is clear that $\mu_t = A$. We must prove that $\mu \in FBT(X)$.

If $x, y \in A$, then $x * y \in A$ and hence $\mu(x) = \mu(y) = \mu(x * y) = t$ which means that

$$\mu(x * y) = t \geq t = T(t, t) = T(\mu(x), \mu(y))$$

and so $\mu \in FBT(X)$.

If $x, y \notin A$, then $\mu(x) = \mu(y) = 0$ which implies that

$$\mu(x * y) \geq 0 = T(0, 0) = T(\mu(x), \mu(y))$$

and then $\mu \in FBT(X)$.

If $x \in A$ and $y \notin A$, then $\mu(x) = t > 0 = \mu(y)$ therefore

$$\mu(x * y) \geq 0 = \mu(y) = T(\mu(x), \mu(y))$$

so $\mu \in FBT(X)$.

Proposition 3.14 Let $\mu_1, \mu_2 \in FBT(X)$. Then $\mu_1 \cap \mu_2 \in FBT(X)$.

Proof 3.15 Let $x, y \in X$. Then

$$\begin{aligned} (\mu_1 \cap \mu_2)(x * y) &= T(\mu_1(x * y), \mu_2(x * y)) \\ &\geq T(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y))) \\ &= T(T(\mu_1(x), \mu_2(x)), T(\mu_1(y), \mu_2(y))) \\ &= T((\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y)) \end{aligned}$$

which means that $\mu_1 \cap \mu_2 \in FBT(X)$.

Example 3.16 Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	1
3	3	0	0	0

Then $(X, *, 0)$ is a B-algebra. Define

$$\mu : X \rightarrow [0, 1]$$

as

$$\mu(x) = \begin{cases} 0.75 & \text{if } x = 0, \\ 0.65 & \text{if } x = 1, \\ 0.55 & \text{if } x = 2, \\ 0.45 & \text{if } x = 3, \end{cases}$$

and

$$\nu : X \rightarrow [0, 1]$$

as

$$\nu(x) = \begin{cases} 0.95 & \text{if } x = 0, \\ 0.6 & \text{if } x = 1, \\ 0.58 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \end{cases}$$

Let $T(a, b) = T_m(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Then $\mu, \nu \in FBT(X)$.

Also

$$\mu \cap \nu : X \rightarrow [0, 1]$$

as

$$(\mu \cap \nu)(x) = T(\mu(x), \nu(x)) = T_m(\mu(x), \nu(x)) = \min\{\mu(x), \nu(x)\} = \begin{cases} 0.75 & \text{if } x = 0, \\ 0.6 & \text{if } x = 1, \\ 0.55 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \end{cases}$$

so $\mu \cap \nu \in FBT(X)$.

Corollary 3.17 *Let $\mu_i \subseteq FBT(X)$ for $i = 1, 2, 3, 4, \dots, n$. Then $\cap_{i=1,2,3,\dots,n} \mu_i \in FBT(X)$.*

Proposition 3.18 *Let $\mu \in FBT(X)$ and T be idempotent t -norm. Then*

$$A = \{x \in X : \mu(x) = \mu(0)\}$$

will be a subalgebra of X .

Proof 3.19 *Let $x, y \in A$ then $\mu(x) = \mu(0) = \mu(y)$. As $\mu \in FBT(X)$ so*

$$\mu(x * y) \geq T(\mu(x), \mu(y)) = T(\mu(0), \mu(0)) = \mu(0) \geq \mu(x * y)$$

thus $\mu(x * y) = \mu(0)$ and then $x * y \in A$ thus A will be a subalgebra of X .

As is well known, the characteristic function of a set is a special fuzzy set. Suppose A is a non-empty subset of X . By χ_A we denote the characteristic function of A , that is,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

Proposition 3.20 *Let T be idempotent t -norm. Then A is a B -subalgebra of X if and only if $\chi_A \in FBT(X)$.*

Proof 3.21 *Let $x, y \in X$.*

(1) *If $x, y \in A$, then $x * y \in A$ and then $\chi_A(x) = \chi_A(y) = \chi_A(x * y) = 1$. Thus*

$$\chi_A(x * y) = 1 \geq 1 = T(1, 1) = T(\chi_A(x), \chi_A(y)).$$

(2) *If $x, y \notin A$, then $\chi_A(x) = \chi_A(y) = 0$. Thus*

$$\chi_A(x * y) \geq 0 = T(0, 0) = T(\chi_A(x), \chi_A(y)).$$

(3) *If $x \in A$ and $y \notin A$, then $\chi_A(x) = 1$ and $\chi_A(y) = 0$. Then*

$$\chi_A(x * y) \geq 0 = T(1, 0) = T(\chi_A(x), \chi_A(y)).$$

(4) *If $x \notin A$ and $y \in A$, then $\chi_A(x) = 0$ and $\chi_A(y) = 1$. So*

$$\chi_A(x * y) \geq 0 = T(1, 0) = T(\chi_A(y), \chi_A(x)) = T(\chi_A(x), \chi_A(y)).$$

Therefore (1)-(4) give us that $\chi_A \in FBT(X)$.

Conversely, let $\chi_A \in FBT(X)$ and $x, y \in A$. Then $\chi_A(x) = \chi_A(y) = 1$ Thus

$$\chi_A(x * y) \geq T(\chi_A(x), \chi_A(y)) = T(1, 1) = 1$$

so $\chi_A(x * y) = 1$ which means that $x * y \in A$ and thus A will be a B -subalgebra of X .

Proposition 3.22 *Let $\mu \in FBT(X)$ and $\nu \in FBT(Y)$. Then $\mu \times \nu \in FBT(X \times Y)$.*

Proof 3.23 *Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then*

$$\begin{aligned}
 (\mu \times \nu)((x_1, y_1) * (x_2, y_2)) &= (\mu \times \nu)(x_1 * x_2, y_1 * y_2) \\
 &= T(\mu(x_1 * x_2), \nu(y_1 * y_2)) \\
 &\geq T(T(\mu(x_1), \mu(x_2)), T(\nu(y_1), \nu(y_2))) \\
 &= T(T(\mu(x_1), \nu(y_1)), T(\mu(x_2), \nu(y_2))) \\
 &= T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2))
 \end{aligned}$$

thus

$$(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) \geq T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2))$$

and so $\mu \times \nu \in FBT(X \times Y)$.

Example 3.24 *Let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:*

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

and $Y = \{0, 1, 2\}$ be a set given by the following Cayley table:

$*$	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

Then $(X, *, 0)$ and $(Y, *, 0)$ will be two B -algebras. Define

$$\mu : X \rightarrow [0, 1]$$

as

$$\mu(x) = \begin{cases} 0.95 & \text{if } x = 0, \\ 0.85 & \text{if } x = 1, \\ 0.15 & \text{if } x = 2, \end{cases}$$

and

$$\nu : Y \rightarrow [0, 1]$$

as

$$\nu(y) = \begin{cases} 0.75 & \text{if } y = 0, \\ 0.4 & \text{if } y = 1, \\ 0.2 & \text{if } y = 2. \end{cases}$$

Let

$$T_{nM}(a, b) = \begin{cases} \min\{a, b\} & \text{if } a + b > 1 \\ 0 & \text{otherwise.} \end{cases}$$

for all $a, b \in [0, 1]$. Then $\mu \in FBT(X)$ and $\nu \in FBT(Y)$. Now

$$\mu \times \nu : X \times Y = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\} \rightarrow [0, 1]$$

as

$$\begin{aligned} (\mu \times \nu)(x, y) &= T(\mu(x), \nu(y)) = T_{nM}(\mu(x), \nu(y)) \\ &= \begin{cases} \min\{\mu(x), \nu(y)\} & \text{if } \mu(x) + \nu(y) > 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0.75 & \text{if } (x, y) = (0, 0) \\ 0.4 & \text{if } (x, y) = (0, 1) \\ 0.2 & \text{if } (x, y) = (0, 2) \\ 0.75 & \text{if } (x, y) = (1, 0) \\ 0.4 & \text{if } (x, y) = (1, 1) \\ 0.2 & \text{if } (x, y) = (1, 2) \\ 0 & \text{if } (x, y) = (2, 0) \\ 0 & \text{if } (x, y) = (2, 1) \\ 0 & \text{if } (x, y) = (2, 2) \end{cases} \end{aligned}$$

thus $\mu \times \nu \in FBT(X \times Y)$.

Proposition 3.25 Let $\mu_i \in FBT(X_i)$, where $1 \leq i \leq n$, then

$$\mu = \prod_{i=1}^n \mu_i \in FBT\left(\prod_{i=1}^n X_i = X\right)$$

such that

$$\mu(x) = \left(\prod_{i=1}^n \mu_i\right)(x_1, x_2, \dots, x_n) = T_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n))$$

for all $x = (x_1, x_2, \dots, x_n) \in X$.

Proposition 3.26 If $\mu \in FBT(X)$ and $\varphi : (X; *, 0) \rightarrow (Y; \dot{*}, \dot{0})$ be an epimorphic B-homomorphism of B-algebras, then $\varphi(\mu) \in FBT(Y)$.

Proof 3.27 Let $x_i \in X$ and $y_i \in Y$ with $\varphi(x_i) = y_i$ and $i = 1, 2$. Then

$$\begin{aligned}\varphi(\mu)(y_1 \dot{*} y_2) &= \sup\{\mu(x_1 * x_2) \mid x_1 * x_2 \in X, \varphi(x_1 * x_2) = y_1 \dot{*} y_2\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_i \in X, \varphi(x_i) = y_i\} \\ &= T(\sup\{\mu(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \sup\{\mu(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\}) \\ &= T(\varphi(\mu)(y_1), \varphi(\mu)(y_2))\end{aligned}$$

thus

$$\varphi(\mu)(y_1 \dot{*} y_2) \geq T(\varphi(\mu)(y_1), \varphi(\mu)(y_2)).$$

Thus $\varphi(\mu) \in FBT(Y)$.

Proposition 3.28 If $\nu \in FBT(Y)$ and $\varphi : (X; *, 0) \rightarrow (Y; \dot{*}, \dot{0})$ be a B -homomorphism of B -algebras, then $\varphi^{-1}(\nu) \in FBT(X)$.

Proof 3.29 Let $x_1, x_2 \in X$.

$$\begin{aligned}\varphi^{-1}(\nu)(x_1 * x_2) &= \nu(\varphi(x_1 * x_2)) \\ &= \nu(\varphi(x_1) * \varphi(x_2)) \\ &\geq T(\nu(\varphi(x_1)), \nu(\varphi(x_2))) \\ &= T(\varphi^{-1}(\nu)(x_1), \varphi^{-1}(\nu)(x_2))\end{aligned}$$

then

$$\varphi^{-1}(\nu)(x_1 * x_2) \geq T(\varphi^{-1}(\nu)(x_1), \varphi^{-1}(\nu)(x_2)).$$

Therefore $\varphi^{-1}(\nu) \in FBT(X)$.

Example 3.30 Let $X = \{0, 1, 2, 3, 4, 5\}$ and $Y = \{0, 1, 2, 3\}$ be two sets given by the following Cayley tables:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

and

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	1
3	3	0	0	0

then $(X, *, 0)$ and $(Y, *, 0)$ will be two B -algebras. Define $\mu : X \rightarrow [0, 1]$ as

$$\mu(x) = \begin{cases} 0.85 & \text{if } x = 0, 1, \\ 0.65 & \text{if } x = 2, 3, \\ 0.55 & \text{if } x = 4, \\ 0.45 & \text{if } x = 5. \end{cases}$$

and $\nu : Y \rightarrow [0, 1]$ as

$$\nu(y) = \begin{cases} 0.75 & \text{if } y = 0, \\ 0.35 & \text{if } y = 1, 2, \\ 0.15 & \text{if } y = 3, \end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ for all $a, b \in [0, 1]$ then $\mu \in FBT(X)$ and $\nu \in FBT(Y)$.

Define B -homomorphism $\varphi : X \rightarrow Y$ as

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, 2, \\ 2 & \text{if } x = 3, 4, \\ 3 & \text{if } x = 5, \end{cases}$$

then we get that $\varphi(\mu) : Y \rightarrow [0, 1]$ as

$$\varphi(\mu)(y) = \sup\{\mu(x) \mid x \in X, \varphi(x) = y\} = \begin{cases} 0.85 & \text{if } y = 0, \\ 0.65 & \text{if } y = 1, \\ 0.55 & \text{if } y = 2, \\ 0.45 & \text{if } y = 3, \end{cases}$$

and thus $\varphi(\mu) \in FBT(Y)$. Also we will have that $\varphi^{-1}(\nu) : X \rightarrow [0, 1]$ as

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)) = \begin{cases} 0.75 & \text{if } x = 0, \\ 0.35 & \text{if } x = 1, 2, 3, 4, \\ 0.15 & \text{if } x = 5, \end{cases}$$

therefore $\varphi^{-1}(\nu) \in FBT(X)$.

4 Fuzzy normal B -algebras under t -norms

Definition 4.1 Let X be a B -algebra. Define $\mu : X \rightarrow [0, 1]$ a fuzzy normal B -algebra under t -norm T if it satisfies the following inequalities:

$$\mu((x * a) * (y * b)) \geq T(\mu(x * y), \mu(a * b))$$

for all $x, y, a, b \in X$.

Denote by $FNBT(X)$, the set of all fuzzy normal B -algebras of B -algebra X under t -norm T .

Example 4.2 Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X, *, 0)$ is a B -algebra. Define $\mu : X \rightarrow [0, 1]$ as

$$\mu(x) = \begin{cases} 0.4 & \text{if } x = 0, 1, \\ 0.5 & \text{if } x = 2, 3. \end{cases}$$

Let $T(a, b) = T_b(a, b) = \max\{0, a + b - 1\}$ for all $a, b \in [0, 1]$ then $\mu \in FNBT(X)$.

Now we prove that every fuzzy normal B -algebra will be fuzzy B -algebra (under t -norm T).

Proposition 4.3 If $\mu \in FBT(X)$. Then $\mu \in FNBT(X)$.

Proof 4.4 Let $x, y \in X$. Then

$$\mu(x * y) = \mu((x * y) * (0 * 0)) \geq T(\mu(x * 0), \mu(y * 0)) = T(\mu(x), \mu(y))$$

then $\mu \in FNBT(X)$.

Remark 4.5 The converse of Proposition 4.3 is not true. For example, let $X = \{0, 1, 2\}$ be a set given by the following Cayley table:

Then $(X, *, 0)$ is a B -algebra. Define $\mu : X \rightarrow [0, 1]$ as

$$\mu(x) = \begin{cases} 0.1 & \text{if } x = 0, \\ 0.2 & \text{if } x = 1, \\ 0.3 & \text{if } x = 2. \end{cases}$$

$*$	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

Let $T(a, b) = T_m(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ then $\mu \in FBT(X)$.
But since

$$\mu((1*2)*(2*1)) = \mu(0) = 0.1T(\mu(1*2), \mu(2*1)) = T(\mu(1), \mu(1)) = T(0.2, 0.2) = 0.2$$

thus $\mu \notin FNBT(X)$.

Proposition 4.6 *Let $\mu \in FNBT(X)$. Then $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$.*

Proof 4.7 *Let $x, y \in X$. Then*

$$\begin{aligned} \mu(x * y) &= \mu((x * y) * (x * x)) \\ &\geq T(\mu(x * x), \mu(y * x)) \\ &= T(\mu(0), \mu(y * x)) \\ &= \mu(y * x) \\ &= \mu((y * x) * (y * y)) \\ &\geq T(\mu(y * y), \mu(x * y)) \\ &= T(\mu(0), \mu(x * y)) \\ &= \mu(x * y) \end{aligned}$$

then $\mu(x * y) = \mu(y * x)$.

Proposition 4.8 *Let $\mu \in FNBT(X)$ and T be idempotent t -norm. Then*

$$N = \{x \in X : \mu(x) = \mu(0)\}$$

will be a normal subalgebra of X .

Proof 4.9 *Let $x, y, a, b \in X$. If $x * y \in N$ and $a * b \in N$, then $\mu(x * y) = \mu(a * b) = \mu(0)$. As $\mu \in FNBT(X)$ so*

$$\mu((x*a)*(y*b)) \geq T(\mu(x*y), \mu(a*b)) = T(\mu(0), \mu(0)) = \mu(0) \geq \mu((x*a)*(y*b))$$

thus $\mu((x * a) * (y * b)) = \mu(0)$ and then $(x * a) * (y * b) \in N$ thus N will be a normal subalgebra of X .

Proposition 4.10 *Let $\mu : X \rightarrow [0, 1]$ be a fuzzy set and T be idempotent t -norm. If $\mu \in FNBT(X)$, then the set*

$$\mu_s = \{x \in X : \mu(x) \geq s\}$$

is either empty or normal subalgebra of B -algebra X for every $s \in [0, 1]$.

Proof 4.11 *Let $\mu \in FNBT(X)$ and $\mu_s = \{x \in X : \mu(x) \geq s\}$ be not empty and $x * y, a * b \in \mu_s$. Then $\mu(x * y) \geq s$ and $\mu(a * b) \geq s$ such that*

$$\mu((x * a) * (y * b)) \geq T(\mu(x * y), \mu(a * b)) \geq T(s, s) = s$$

which means that $(x * a) * (y * b) \in \mu_s$. Hence μ_s is normal subalgebra of B -algebra X for every $s \in [0, 1]$, which proves the proposition.

Proposition 4.12 *Let $\mu \in FNBT(X)$ and $\nu \in FNBT(X)$. Then $\mu \cap \nu \in FNBT(X)$.*

Proof 4.13 *Let $x, y, a, b \in X$. As*

$$\begin{aligned} (\mu \cap \nu)((x * a) * (y * b)) &= T(\mu((x * a) * (y * b)), \nu((x * a) * (y * b))) \\ &\geq T(T(\mu(x * y), \mu(a * b)), T(\nu(x * y), \nu(a * b))) \\ &= T(T(\mu(x * y), \nu(x * y)), T(\nu(a * b), \nu(a * b))) \\ &= T((\mu \cap \nu)(x * y), (\mu \cap \nu)(a * b)) \end{aligned}$$

so

$$(\mu \cap \nu)((x * a) * (y * b)) \geq T((\mu \cap \nu)(x * y), (\mu \cap \nu)(a * b)).$$

Then $\mu \cap \nu \in FNBT(X)$.

Proposition 4.14 *Let $\mu \in FNBT(X)$ and $\nu \in FNBT(Y)$. Then $\mu \times \nu \in FNBT(X \times Y)$.*

Proof 4.15 *Let $(x_1, x_2), (y_1, y_2), (a_1, a_2), (b_1, b_2) \in X \times Y$. Then*

$$\begin{aligned} (\mu \times \nu)((x_1, x_2) * (a_1, a_2)) * ((y_1, y_2) * (b_1, b_2)) \\ &= (\mu \times \nu)((x_1 * a_1, x_2 * a_2) * (y_1 * b_1, y_2 * b_2)) \\ &= (\mu \times \nu)((x_1 * a_1) * (y_1 * b_1), (x_2 * a_2) * (y_2 * b_2)) \\ &= T(\mu((x_1 * a_1) * (y_1 * b_1)), \nu((x_2 * a_2) * (y_2 * b_2))) \\ &\geq T(T(\mu(x_1 * y_1), \mu(a_1 * b_1)), T(\nu(x_2 * y_2), \nu(a_2 * b_2))) \end{aligned}$$

$$\begin{aligned}
 &= T(T(\mu(x_1 * y_1), \nu(x_2 * y_2)), T(\mu(a_1 * b_1), \nu(a_2 * b_2))) \\
 &= T((\mu \times \nu)(x_1 * y_1, x_2 * y_2), (\mu \times \nu)(a_1 * b_1, a_2 * b_2)) \\
 &= T((\mu \times \nu)((x_1, x_2) * (y_1, y_2)), (\mu \times \nu)((a_1, a_2) * (b_1, b_2)))
 \end{aligned}$$

and so

$$\begin{aligned}
 &(\mu \times \nu)((x_1, x_2) * (a_1, a_2)) * ((y_1, y_2) * (b_1, b_2)) \\
 &\geq T((\mu \times \nu)((x_1, x_2) * (y_1, y_2)), (\mu \times \nu)((a_1, a_2) * (b_1, b_2))).
 \end{aligned}$$

Thus $\mu \times \nu \in FNBT(X \times Y)$.

Proposition 4.16 *Let $\mu_i \in FNBT(X_i)$, where $1 \leq i \leq n$, then*

$$\mu = \prod_{i=1}^n \mu_i \in FNBT(\prod_{i=1}^n X_i = X)$$

such that

$$\mu(x) = (\prod_{i=1}^n \mu_i)(x_1, x_2, \dots, x_n) = T_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n))$$

for all $x = (x_1, x_2, \dots, x_n) \in X$.

Proposition 4.17 *If $\mu \in FNBT(X)$ and $\varphi : (X; *, 0) \rightarrow (Y; \dot{*}, \dot{0})$ be an epimorphic B-homomorphism of B-algebras, then $\varphi(\mu) \in FNBT(Y)$.*

Proof 4.18 *Let $x_i \in X$ and $y_i \in Y$ with $\varphi(x_i) = y_i$ and $i = 1, 2, 3, 4$. Then*

$$\begin{aligned}
 \varphi(\mu)((y_1 \dot{*} y_2) \dot{*} (y_3 \dot{*} y_4)) &= \sup\{(\mu(x_1 * x_2) * (x_3 * x_4)) \mid x_i \in X, \varphi(x_i) = y_i\} \\
 &\geq \sup\{T(\mu(x_1 * x_3), \mu(x_2 * x_4)) \mid x_i \in X, \varphi(x_i) = y_i\} \\
 &= T(\sup\{\mu(x_1 * x_3) \mid \varphi(x_i) = y_i\}, \sup\{\mu(x_2 * x_4) \mid \varphi(x_i) = y_i\}) \\
 &= T(\varphi(\mu)((y_1 \dot{*} y_3)), \varphi(\mu)((y_2 \dot{*} y_4)))
 \end{aligned}$$

thus

$$\varphi(\mu)((y_1 \dot{*} y_2) \dot{*} (y_3 \dot{*} y_4)) \geq T(\varphi(\mu)((y_1 \dot{*} y_3)), \varphi(\mu)((y_2 \dot{*} y_4))).$$

Thus $\varphi(\mu) \in FNBT(Y)$.

Proposition 4.19 *If $\nu \in FNBT(Y)$ and $\varphi : (X; *, 0) \rightarrow (Y; \dot{*}, \dot{0})$ be a B-homomorphism of B-algebras, then $\varphi^{-1}(\nu) \in FNBT(X)$.*

Proof 4.20 Let $x_1, x_2, x_3, x_4 \in X$. Now

$$\begin{aligned}
 \varphi^{-1}(\nu)((x_1 * x_2) * (x_3 * x_4)) &= \nu(\varphi((x_1 * x_2) * (x_3 * x_4))) \\
 &= \nu(\varphi(x_1 * x_2) * \varphi(x_3 * x_4)) \\
 &= \nu((\varphi(x_1) * \varphi(x_2)) * (\varphi(x_3) * \varphi(x_4))) \\
 &\geq T(\nu(\varphi(x_1) * \varphi(x_3)), \nu(\varphi(x_2) * \varphi(x_4))) \\
 &= T(\nu(\varphi(x_1 * x_3)), \nu(\varphi(x_2 * x_4))) \\
 &= T(\varphi^{-1}(\nu)(x_1 * x_3), \varphi^{-1}(\nu)(x_2 * x_4))
 \end{aligned}$$

then

$$\varphi^{-1}(\nu)((x_1 * x_2) * (x_3 * x_4)) \geq T(\varphi^{-1}(\nu)(x_1 * x_3), \varphi^{-1}(\nu)(x_2 * x_4)).$$

Therefore $\varphi^{-1}(\nu) \in FNBT(X)$.

5 Open Problem

In this paper, we introduced the concepts of fuzzy B -algebras and normal fuzzy B -algebra with respect to t -norms and we considered the properties of them. Now one can introduce and investigate fuzzy BCH -algebras and normal fuzzy BCH -algebras with respect to t -norms as we did and this can be an open problem.

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