Int. J. Open Problems Compt. Math., Vol. 16, No. 1, March 2023
Print ISSN: 1998-6262, Online ISSN: 2079-0376
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Inequalities of bi-starlike functions involving Sigmoid function and Bernoulli Lemniscate by subordination

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Received 15 February 2023; Accepted 25 March 2023

Abstract

The sigmoid function increases the size of the hypothesis space that the network can represent. Neural networks can be used for complex learning tasks. It is therefore necessary to investigate the role of sigmoid function in geometric function theory. In this study, a new subclass of bi-starlike functions involving Sigmoid function and Bernolli Lemniscate was defined. Some coefficient bounds belonging to this newly defined subclass were also obtained by using subordination principle. The key tools in the proof of our main results are the coefficient Fekete-Szegö inequalities for this subclass. The results obtained agree and extend some earlier results.

Keywords: *Bernolli Lemniscate, Bi-starlike functions, Sigmoid function, Subordination, Univalent functions.*

1 Introduction

Special functions deal with an information process that is inspired by the way nervous system such as brain processes information. It comprises of large number of highly interconnected processing elements (neurones) working together

to solve a specific problem. The functions are overshadow by different fields as real analysis, algebra, topology, functional analysis, differential equations and so on because it imitates the way human brain works. They can be programmed to solve a particular problem and it can also be trained by samples.

Special functions can be categorized into three namely, threshold function, ramp function and the logistic sigmoid function. The most important one among all is the logistic sigmoid function. The most important one among all is the logistic sigmoid function because of its gradient descendent learning algorithm. It can be evaluated in different ways, most especially by truncated series expansion. The logistic sigmoid function of the form

$$h(s) = \frac{1}{1+e^{-s}}, \qquad s \geq 0, \tag{1}$$

is differentiable and has the following properties:

(i) It outputs real numbers between 0 and 1.

(ii) It maps a very large input domain to a small range of outputs.

(iii)It never loses information because it is a one-to-one function.

(iv) It increases monotonically.

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With all the mentioned properties above, it is clear that logistic sigmoid function is very useful in geometric functions theory ([2], [5], [9], [10], [11]).

Let $\mathcal{A}$ be the family of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

which are analytic in the open unit disc $\mathfrak{U} = \{z : |z| < 1\}$ and normalized under the conditions given by $f(0) = 0 = f'(0)-1$. Let $S = \{f \in \mathcal{A} : f \text{ is univalent in } \mathfrak{U}\}$.Recall

that, S^* and K are the two usual classes of starlike and convex functions which their geometric conditions satisfies $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0$

According to the Koebe One-Quarter Theorem [4] every function $f \in S$ has an inverse f^{-1} which satisfies the following conditions:

$$f^{-1}\left(f\left(z\right)\right) = z \quad (z \in \mathfrak{U})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right), \quad (w \in \mathfrak{U})$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots . \quad (3)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both f and f^{-1} are univalent in $\mathfrak{U}$. Let Σ denote the class of bi-univalent functions in $\mathfrak{U}$ given by (2). Many researchers have investigated several interesting special families of Σ(see,[3, 7, 8, 14])

Two analytic functions are said to be subordinate to each other written as $f \prec g$, if there exists a Schwartz function $w(z)$ which is analytic in $\mathfrak{U}$ with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in \mathfrak{U}$, such that $f(z) = g(w(z))$, and $f(\mathfrak{U}) \subset g(\mathfrak{U})$ ([13]).

Sokol and Thomas [15] introduced and studied the class S_L^* in the unit disc $\mathfrak{U}$, normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z} =: q(z), \quad z \in \mathfrak{U}, \quad (4)$$

where the branch of the square root is choosen to be $q(0) = 1$.
It also noted that the set $q(U)$ lies in the region bounded by the right loop of the Lemniscate of Bernolli $\gamma_1 : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.
Fadipe-Joseph et al. ([5]) studied the modified sigmoid function

$$G(z) = \frac{2}{1 + e^{-z}}$$

and obtained another series of the modified sigmoid function as

$$G(z) = 1 + \left[\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\frac{(-1)^n}{n!}z^n\right)^m\right]$$

$$= 1 + \frac{1}{2}z + \frac{1}{24}z^3 + \frac{1}{240}z^5 + \cdots \quad .$$

Consider the function

$$f_\gamma(z) = z + \sum_{n=2}^{\infty} \gamma(s)a_nz^n, \quad (5)$$

$$\gamma(s) = \frac{2}{1 + e^{-s}} \qquad s \geq 0.$$

Functions of the form (5) belong to the class A_γ, where $A_1 \equiv A$.
Motivation by ([1], [5], [6], [9], [12] and [16]), new subclasses were introduced for bi-univalent functions related to Bernoulli lemniscate and modified sigmoid function. The coefficient bounds and Fekete-Szegö inequality for two subclasses defined were obtained. The results are new and generates many corollaries.

For the main purpose of our discussion, we shall give the some lemmas and definitions as follow:

Lemma 1.1 *(see [13]) If a function $p \in P$ is given by $p(z) = 1 + p_1 z + p_2 z^2 + \cdots (z \in \mathfrak{U})$, then $|p_k| \leq 2, \quad n \in \mathbb{N}$, where P is the family of all functions analytic in $\mathfrak{U}$ for which $p(0) = 1$ and $Re(p(z)) > 0, \quad (z \in \mathfrak{U})$.*

Lemma 1.2 *(see [13]) If a function $p \in P$ is given by $p(z) = 1 + p_1 z + p_2 z^2 + \cdots (z \in \mathfrak{U})$, then $|p_k| \leq 2, \quad n \in \mathbb{N}$, where P is the family of all functions analytic in $\mathfrak{U}$ for which $p(0) = 1$ and $Re(p(z)) > 0, \quad (z \in \mathfrak{U})$.*

Lemma 1.3 *(see [5]) Let h be a sigmoid function and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m, \tag{6}$$

then $\Phi(z) \in P \quad |z| < 1$, where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.4 *(see [5]) Let*

$$\Phi_{m,n}(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right), \tag{7}$$

then $|\Phi_{m,n}(z)| < 2$.

Lemma 1.5 *(see [5]) If $\Phi(z) \in P$ and it is starlike, then f is normalized univalent function of the form (2).*

Setting $m = 1$, Fadipe-Joseph et al. [5] remarked that

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \frac{(-1)^{n+1}}{2n!}$. As such, $|c_n| \leq 2, \quad n = 1, 2, 3, \ldots$ and the result is sharp for each n.

Definition 1.6 *A function $f \in \Sigma$ is said to be in the class $\mathfrak{H}_{\Sigma}^{\gamma}(\lambda)$ if and only if*

$$\left| \left[\frac{z[f'_{\gamma}(z)]^{\lambda}}{f_{\gamma}(z)} \right]^{2} - 1 \right| < 1 \qquad (z \in \mathfrak{U}). \tag{8}$$

Equivalently, from (8) and definition of subordination that a function $f \in \mathfrak{H}_{\Sigma}^{\gamma}(\lambda)$ fulfills the condition of subordination given below:

$$\frac{z[f'_{\gamma}(z)]^{\lambda}}{f_{\gamma}(z)} \prec \sqrt{1+z} \qquad (z \in \mathfrak{U}). \tag{9}$$

and

$$\left| \left[\frac{w[f'_{\gamma}(w)]^{\lambda}}{f_{\gamma}(w)} \right]^{2} - 1 \right| < 1 \qquad (w \in \mathfrak{U}). \tag{10}$$

Equivalently, from (10) and definition of subordination that a function $f \in \mathfrak{H}_{\Sigma}^{\gamma}(\lambda)$ fulfills the condition of subordination given below:

$$\frac{w[f'_{\gamma}(w)]^{\lambda}}{f_{\gamma}(w)} \prec \sqrt{1+w} \qquad (w \in \mathfrak{U}). \tag{11}$$

where g is an extension of $f^{-1} \in \mathfrak{U}$.

We further note that if $f \in \mathfrak{H}_{\Sigma}^{\gamma}(\lambda)$, then the function $\frac{z[f'_{\gamma}(z)]^{\lambda}}{f_{\gamma}(z)}$ lies in the region bounded by the right half of lemniscate of Bernolli given by

$$\left\{ \varpi \in \mathcal{C} : |\varpi^{2} - 1| < 1 \right\} = \left\{ x + iy : (x^{2} + y^{2})^{2} - 2(x^{2} - y^{2}) = 0 \right\}. \tag{12}$$

Specializing the parameter $\lambda = 1$, we have the following definition

Definition 1.7 *A function $f \in \Sigma$ is said to be in the class $\mathfrak{H}_{\Sigma}^{\gamma}$ if and only if*

$$\left| \left[\frac{z[f'_{\gamma}(z)]}{f_{\gamma}(z)} \right]^{2} - 1 \right| < 1 \qquad (z \in \mathfrak{U}). \tag{13}$$

Equivalently, we have

$$\frac{z[f'_{\gamma}(z)]}{f_{\gamma}(z)} \prec \sqrt{1+z} \qquad (z \in \mathfrak{U}). \tag{14}$$

and

$$\left| \left[\frac{w[f'_{\gamma}(w)]}{f_{\gamma}(w)} \right]^{2} - 1 \right| < 1 \qquad (w \in \mathfrak{U}). \tag{15}$$

Equivalently, we have

$$\frac{w[f'_{\gamma}(w)]}{f_{\gamma}(w)} \prec \sqrt{1+w} \qquad (w \in \mathfrak{U}). \tag{16}$$

where g is an extension of $f^{-1} \in \mathfrak{U}$.

2 Main Results

Theorem 2.1 *Let $f \in \mathfrak{H}_{\Sigma}^{\gamma}(\lambda)$ and be of the form (2). Then,*

$$|a_2| \leq \frac{1}{\gamma(s)}\sqrt{\frac{3}{2(2\lambda-1)(163\lambda-80)}}, \tag{17}$$

and

$$|a_3| \leq \frac{1}{2\gamma(s)}\left[\frac{1}{3\lambda-1}+\frac{3}{(2\lambda-1)(163\lambda-80)}\right]. \tag{18}$$

Proof. If $f \in \mathfrak{H}_{\Sigma}^{\gamma}(\lambda)$, then it follows:

$$\frac{z[f_{\gamma}'(z)]^{\lambda}}{f_{\gamma}(z)} \prec \phi(z), \qquad where \qquad \phi(z)=\sqrt{1+z}. \tag{19}$$

Define a function

$$p(z)=\frac{1+\varphi(z)}{1-\varphi(z)}=1+p_1z+p_2z^2+\dots \qquad .$$

It is clear that $p \in P$. This implies that

$$\varphi(z)=\frac{p(z)-1}{p(z)+1}, \qquad from \quad (19) \quad we \quad have$$

$$\frac{z[f_{\gamma}'(z)]^{\lambda}}{f_{\gamma}(z)}=\phi(\varphi(z)), \qquad with \qquad \phi(\varphi(z))=\left(\frac{2p(z)}{p(z)+1}\right)^{\frac{1}{2}}. \tag{20}$$

Now,

$$\left(\frac{2p(z)}{p(z)+1}\right)^{\frac{1}{2}}=1+\frac{1}{4}p_1z+\left[\frac{1}{4}p_2-\frac{5}{32}p_1^2\right]z^2+\left[\frac{1}{4}p_3-\frac{5}{16}p_1p_2+\frac{13}{128}p_1^3\right]z^3+\dots \qquad . \tag{21}$$

Similarly,

$$\frac{w[g_{\gamma}'(w)]^{\lambda}}{g_{\gamma}(w)} \prec \phi(w), \qquad where \qquad \phi(w)=\sqrt{1+w}. \tag{22}$$

Define a function

$$q(w)=\frac{1+\varphi(w)}{1-\varphi(w)}=1+q_1z+q_2z^2+\dots \qquad .$$

It is clear that $q \in P$. This implies that

$$\varphi(w)=\frac{q(w)-1}{q(w)+1}, \qquad from(22) \quad we \quad have$$

$$\frac{w[g'_\gamma(w)]^\lambda}{g_\gamma(w)} = \phi(\varphi(w)), \quad with \quad \phi(\varphi(w)) = \left(\frac{2q(w)}{q(w)+1}\right)^{\frac{1}{2}}. \tag{23}$$

Now,

$$\left(\frac{2q(w)}{q(w)+1}\right)^{\frac{1}{2}} = 1+\frac{1}{4}q_1w+\left[\frac{1}{4}q_2-\frac{5}{32}q_1^2\right]w^2+\left[\frac{1}{4}q_3-\frac{5}{16}q_1q_2+\frac{13}{128}q_1^3\right]w^3+\dots \quad . \tag{24}$$

Also,

$$\frac{z[f'_\gamma(z)]^\lambda}{f_\gamma(z)} = 1+(2\lambda-1)\gamma(s)a_2z+\left[(3\lambda-1)\gamma(s)a_3+(2\lambda^2-4\lambda+1)\gamma^2(s)a_2^2\right]z^2+\dots \quad , \tag{25}$$

and

$$\frac{w[g'_\gamma(w)]^\lambda}{g_\gamma(w)} = 1-(2\lambda-1)\gamma(s)a_2w+\left[(2\lambda^2+2\lambda-1)\gamma^2(s)a_2^2-(3\lambda-1)\gamma(s)a_3\right]w^2+\dots \quad . \tag{26}$$

Now, equating the coefficients in (20) and (23), we have

$$4(2\lambda-1)\gamma(s)a_2 = p_1 \tag{27}$$

$$(3\lambda-1)\gamma(s)a_3+(2\lambda^2-4\lambda+1)\gamma^2(s)a_2^2 = \frac{1}{4}p_2-\frac{5}{32}p_1^2 \tag{28}$$

$$-4(2\lambda-1)\gamma(s)a_2 = q_1 \tag{29}$$

$$-(3\lambda-1)\gamma(s)a_3+(2\lambda^2+2\lambda-1)\gamma^2(s)a_2^2 = \frac{1}{4}q_2-\frac{5}{32}q_1^2. \tag{30}$$

From equations (27) and (29) we have

$$p_1 = -q_1 \tag{31}$$

$$32(2\lambda-1)^2\gamma^2(s)a_2^2 = p_1^2+q_1^2. \tag{32}$$

So,

$$a_2^2 = \frac{p_1^2+q_1^2}{32(2\lambda-1)^2\gamma^2(s)}. \tag{33}$$

Hence applying Lemma 1.1 for the coefficient p_1 and q_1

$$|a_2| \le \frac{1}{2(2\lambda-1)\gamma(s)}. \tag{34}$$

Now, by adding equations (28) and (30), we get

$$2\lambda(2\lambda-1)\gamma^2(s)a_2^2 = \frac{1}{4}(p_2+q_2)-\frac{5}{3}(p_1^2+q_1^2). \tag{35}$$

Substituting (32) in (35), we have

$$2\lambda(2\lambda-1)\gamma^2(s)a_2^2=\frac{1}{4}(p_2+q_2)-\frac{5}{3}\left[32(2\lambda-1)^2\gamma^2(s)a_2^2\right] \tag{36}$$

$$2a_2^2\gamma^2(s)[(2\lambda-1)(163\lambda-80)]=\frac{3}{4}(p_2+q_2)$$

$$8a_2^2\gamma^2(s)[(2\lambda-1)(163\lambda-80)]=3(p_2+q_2). \tag{37}$$

Therefore by Lemma 1.1, we get

$$|a_2|\leq\sqrt{\frac{3}{2(2\lambda-1)(163\lambda-80)\gamma^2(s)}}. \tag{38}$$

It is clear that

$$min\left\{\frac{1}{2(2\lambda-1)\gamma(s)},\quad\frac{1}{\gamma(s)}\sqrt{\frac{3}{2(2\lambda-1)(163\lambda-80)}}\right\}.$$

So, we obtain the desired inequality in Theorem 2.1.
Next, in order to find the bound on $|a_3|$, by subtracting (30) from (28), we get

$$2(3\lambda-1)\gamma(s)a_3+2(1-3\lambda)\gamma^2(s)a_2^2=\frac{1}{4}(p_2-q_2)-\frac{5}{32}(p_1^2-q_1^2). \tag{39}$$

From (31), we know that $p_1^2=q_1^2$ and also using (37), we obtain

$$2(3\lambda-1)\gamma(s)a_3+2(1-3\lambda)\gamma^2(s)\left[\frac{3(p_2+q_2)}{8\gamma^2(s)(2\lambda-1)(163\lambda-80)}\right]=\frac{1}{4}(p_2-q_2).$$

Applying Lemma 1.1 for the coefficient p_2 and q_2 and taking modulus of a_3,

$$2(3\lambda-1)\gamma(s)a_3=\frac{1}{4}(p_2-q_2)+2(3\lambda-1)\left[\frac{3(p_2+q_2)}{8(2\lambda-1)(163\lambda-80)}\right]$$

$$|a_3|\leq\frac{1}{2\gamma(s)}\left[\frac{1}{3\lambda-1}+\frac{3}{(2\lambda-1)(163\lambda-80)}\right].$$

Hence,the proof is completed.

In 1933, Fekete and Szegö gave the sharp bound for the function $|a_3-\mu a_2^2|$ for the class S of univalent functions when μ is real. The determination of the sharp bounds for the functional $|a_3-\mu a_2^2|$ is known as the relevant connection to the Fekete-Szegö inequality and this has been studied by many researchers for different subclasses of univalent functions[6].

The next theorem gives us the Fekete-Szegö inequality :

Theorem 2.2 *Let f given by (2) be in the class $\mathfrak{H}_{\Sigma}^{\gamma}(\lambda,\mu)$ and $\mu \in \mathbb{R}$. Then we have*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{(3\lambda-1)\gamma(s)} & , \quad 0 \leq |h(\mu)| \leq \frac{1}{8(3\lambda-1)\gamma(s)}, \\ 8\left|h(\mu)\right| & , \quad |h(\mu)| \geq \frac{1}{8(3\lambda-1)\gamma(s)}, \end{cases} \tag{40}$$

where

$$h(\mu)=\frac{3(\gamma(s)-\mu)}{8(2\lambda-1)(163\lambda-80)\gamma^2(s)}. \tag{41}$$

Proof. From the equations (33) and (35), we get

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{(p_2-q_2)(2\lambda-1)(163\lambda-80)+3(3\lambda-1)(p_2+q_2)}{8(2\lambda-1)(3\lambda-1)(163\lambda-80)\gamma(s)} - \mu\frac{3(p_2+q_2)}{8\gamma^2(s)(2\lambda-1)(163\lambda-80)} \\ &= \frac{p_2-q_2}{8(3\lambda-1)\gamma(s)}+\frac{3(p_2+q_2)(\gamma(s)-\mu)}{8(2\lambda-1)(163\lambda-80)\gamma^2(s)} \\ &= \left[\frac{3(\gamma(s)-\mu)}{8(2\lambda-1)(163\lambda-80)\gamma^2(s)}+\frac{1}{8(3\lambda-1)\gamma(s)}\right]p_2 \\ &\qquad + \left[\frac{3(\gamma(s)-\mu)}{8(2\lambda-1)(163\lambda-80)\gamma^2(s)}-\frac{1}{8(3\lambda-1)\gamma(s)}\right]q_2. \end{aligned}$$

So, we have

$$a_3-\mu a_2^2=\left[h(\mu)+\frac{1}{8(3\lambda-1)\gamma(s)}\right]p_2+\left[h(\mu)-\frac{1}{8(3\lambda-1)\gamma(s)}\right]q_2, \tag{42}$$

where

$$h(\mu)=\frac{3(\gamma(s)-\mu)}{8(2\lambda-1)(163\lambda-80)\gamma^2(s)}.$$

Then, by taking modulus of (42), we conclude the proof.

3 Corollaries and Consequences

An urgent and important corollaries of the Theorem 2.1 and Theorem 2.2 for $s=0$ (it is clear that $\gamma(0)=1$) is asserted by Corollary 3.1 and Corollary 3.2 respectively.

Corollary 3.1 *Let $f \in \mathfrak{H}_{\Sigma}^{1}(\lambda)$. Then,*

$$|a_2| \leq \sqrt{\frac{3}{2(2\lambda-1)(163\lambda-80)}}, \tag{43}$$

and

$$|a_3| \leq \frac{1}{2}\left[\frac{1}{3\lambda-1}+\frac{3}{(2\lambda-1)(163\lambda-80)}\right]. \tag{44}$$

Corollary 3.2 *Let f given by (2) be in the class $\mathfrak{H}_{\Sigma}^{1}(\lambda,\mu)$ and $\mu \in \mathbb{R}$. Then we have*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{(3\lambda-1)} & , \quad 0 \leq |h(\mu)| \leq \frac{1}{8(3\lambda-1)}, \\ 8\,|h(\mu)| & , \quad |h(\mu)| \geq \frac{1}{8(3\lambda-1)}, \end{cases} \tag{45}$$

where

$$h(\mu)=\frac{3(1-\mu)}{8(2\lambda-1)(163\lambda-80)}. \tag{46}$$

In particular, choosing $\lambda = 1$, in Corollary 3.1 and Corollary 3.2, we get following corollaries respectively:

Corollary 3.3 *Let $f \in \mathfrak{H}_{\Sigma}^{1}(1)$. Then,*

$$|a_2| \leq \sqrt{\frac{3}{166}}, \tag{47}$$

and

$$|a_3| \leq \frac{89}{332}. \tag{48}$$

Corollary 3.4 *Let f given by (2) be in the class $\mathfrak{H}_{\Sigma}^{1}(1,\mu)$ and $\mu \in \mathbb{R}$. Then we have*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{2} & , \quad 0 \leq |h(\mu)| \leq \frac{1}{16}, \\ 8\,|h(\mu)| & , \quad |h(\mu)| \geq \frac{1}{16}, \end{cases} \tag{49}$$

where

$$h(\mu)=\frac{3(1-\mu)}{664}. \tag{50}$$

In particular, for $\mu = 1$ in corollary 3.2 and corollary 3.4, we have following corallaries respectively:

Corollary 3.5 *Let f given by (2) be in the class $\mathfrak{H}_{\Sigma}^{1}(\lambda,1)$. Then we have*

$$\left|a_3-a_2^2\right| \leq \frac{1}{(3\lambda-1)}. \tag{51}$$

Corollary 3.6 *Let f given by (2) be in the class $\mathfrak{H}_{\Sigma}^{1}(1,1)$. Then we have*

$$\left|a_3-a_2^2\right| \leq \frac{1}{2} \tag{52}$$

4 Conclusion

In this current work, a new subclass $\mathfrak{H}_{\Sigma}^{\gamma}(\lambda)$ of bi-starlike functions with the help of Sigmoid function and Bernolli Lemniscate was determined. Coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö inequalities belonging to this subclass were also obtained by using subordination principle. The results obtained extended to some previous results as a consequences.

5 Open Problem

As an open problem, we can point out the following:

Firstly, Hankel determinant for this defined class can be investigated by other researchers.

Secondly, geometric properties of Sigmoid function can be examined due to its novelty in literature.

Lastly, radii of starlikeness associated with the Lemniscate of Bernoulli and the left-half plane can be investigated in this class.

We hope that this work encourage the researchers to obtain other characterization properties and relevant connections in other classes of univalent functions.

Conflict of interest. All authors declare that there is not any conflict of interests concerning the publication of this manuscript.

ACKNOWLEDGEMENTS. The authors thank the referees for their insightful suggestions to improve the paper in present form.

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