

RESEARCH ARTICLE

An Explicit Forward-Euler Type Method for Time-Space Fractional Advection Equations

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ABSTRACT

This paper introduces an explicit finite-difference scheme (FDS) for time–space fractional-order (FO) advection equations whose temporal and spatial derivatives have different FOs. The discretization is constructed from a FO Taylor expansion and produces a forward-Euler–type approximation of Caputo derivatives. We provide a rigorous treatment of the scheme’s consistency, establish stability via Fourier analysis, and derive convergence by combining stability with local truncation error estimates. Numerical experiments corroborate the theoretical findings, showing the method efficiently captures the nonlocal memory effects of FO models. The approach is robust and readily extendable to multidimensional and nonlinear settings.

Keywords: Fractional advection equations; explicit finite difference scheme; Caputo derivative; stability analysis.

Mathematics Subject Classification: 65M06, 65M12, 35R11, 26A33 (MSC2020).

1 Introduction

Over the past decades, FO calculus which extends the notion of differentiation and integration from integer orders to FO ones has attracted considerable attention across scientific and engineering communities [1,2]. This growing interest arises from the recognition that many complex processes can be described more accurately using FO-PDEs rather than their integer order counterparts [3,4]. FO-PDEs are particularly well suited for modeling phenomena with long-term memory and hereditary characteristics, including viscoelasticity [5], anomalous diffusion in porous media [6,7] and various applications in signal processing and control theory [8]. The inherent nonlocality of FO operators where the present state depends on the full history of the system makes them especially powerful for capturing such dynamics. Among these equations FO advection diffusion models play a crucial role in describing transport in heterogeneous media where particle motion deviates from classical Brownian dynamics [9–11]. In this context, the advection term represents bulk transport, while FO diffusion or dispersion accounts for anomalous spreading either subdiffusive or super diffusive. The present study focuses on a particular case: a pure FO advection equation characterized by distinct fractional orders in time and space.

The search for closed form solutions for FO-PDEs is normally restricted to highly idealized conditions, making it extremely difficult for analytical solutions [12, 13]. As a result, the need for efficient numerical methods has been emphasized. Various numerical methods have been tried for solving these equations, including the Adomian Decomposition Method (ADM) [14, 15], Variational Iteration Method (VIM) [16–19], Finite Element Methods (FEMs) [20–22], and Spectral Methods [23–27]. Despite these advances the FDS remains one of the most widely used approaches due to its conceptual simplicity and ease of implementation [28, 29]. For time fractional equations, many FDMs rely on the L1 discretization of the Caputo derivative [30, 31]. An implicit L1 method was proposed by Langlands and Henry and its stability and accuracy were proved [32]. In the case when the space FO operators are involved, the shifted

Grünwald–Letnikov formula is used [33]. In order to increase the accuracy of the solutions, compact difference schemes and Crank–Nicolson schemes have also been implemented [34–37].

Stability and convergence analyses remain central when assessing the reliability of these discretizations. For explicit schemes applied to linear FO-PDEs with constant coefficients von Neumann (Fourier) analysis is a standard tool [38] while implicit approaches are typically studied using energy based arguments to ensure unconditional stability [39,40]. Although implicit schemes offer greater stability margins explicit formulations are often more attractive due to their lower computational cost per iteration albeit subject to restrictions on the relationship between spatial and temporal step sizes [41,42].

In this paper we introduce a straightforward explicit FDS for a FO advection equation with distinct time and space FOs. The derivation employs a fractional Taylor expansion leading to a forward Euler type discretization in both variables. A comprehensive theoretical investigation is conducted covering consistency, stability and convergence properties of the method. Numerical experiments are then carried out to validate the analytical results and to demonstrate the scheme's effectiveness. The remainder of this work is organized as follows: Section 2 presents the governing FO advection model, the discretization framework, and the derivation of the proposed explicit FDS. Section 3 discusses the theoretical analysis, and Section 4 includes the numerical experiments. Section 5 summarizes the key outcomes and highlights possible future directions.

2 Problem Formulation

In this study, we address the time–FO advection problem described by

$$\frac{\partial^{\varrho} u}{\partial t^{\varrho}}(x, t) = \frac{\partial^{\sigma} u}{\partial x^{\sigma}}(x, t) + f(x, t), \quad x \in \Omega, \quad 0 < t < T, \quad \varrho, \sigma \in (0, 1). \quad (1)$$

The system is supplemented with the PCs and ICs

$$\left. \frac{\partial u}{\partial x} \right|_{\partial \Omega} = 0, \quad 0 < t < T, \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (3)$$

$$u(x, T) = u_1(x), \quad x \in \Omega. \quad (4)$$

In this model, $u = u(x, t)$ represents the concentration field, and the FO derivatives $\frac{\partial^{\varrho} u}{\partial t^{\varrho}}$ and $\frac{\partial^{\sigma} u}{\partial x^{\sigma}}$ are defined in the Caputo sense.

Definition 2.1 ([43]). Let $u(x, t)$ be locally integrable on $\mathbb{R}_+$. The Riemann–Liouville FO integral is given by

$$I_t^{\varrho} u(x, t) = \frac{1}{\Gamma(\varrho)} \int_0^t (t - \tau)^{\varrho-1} u(x, \tau) d\tau, \quad \varrho > 0. \quad (5)$$

Definition 2.2 ([44]). The Caputo FO derivative is given by

$$\frac{\partial^{\varrho} u(x, t)}{\partial t^{\varrho}} = \frac{1}{\Gamma(1 - \varrho)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} (t - \tau)^{-\varrho} d\tau, \quad \forall (\varrho, t) \in (0, 1) \times \mathbb{R}_+^*. \quad (6)$$

The above equation is an example of the contemporary generalization of the classical transport theories, based on the FO calculus, which is a branch of mathematical analysis that generalizes the concepts of differentiation and integration to FOs. The evolution of the FO calculus from the early hints in the correspondence between Leibniz and l'Hôpital (1695) to the rigorous definitions by Riemann, Liouville, and later Caputo, is the required mathematical foundation for the above theories. The Caputo derivative is the most suitable for physical problems because it naturally includes standard initial conditions.

The integer-order counterpart of (1), obtained when $\varrho = \sigma = 1$, is the classical advection Eq. (7):

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0, \quad (7)$$

which describes the transport of a conserved quantity at constant velocity. For centuries, this model and its extensions (e.g., the advection-diffusion equation) successfully described Fickian transport, where the mean squared displacement grows linearly with time. However, experimental observations across various fields (e.g., physics, hydrology, biology) revealed anomalous transport phenomena that deviate from this classical behavior:

- **Super-diffusion:** $\langle x^2 \rangle \sim t^\gamma$, $\gamma > 1$, observed in turbulent flows or porous media with preferential paths.
- **Sub-diffusion:** $\langle x^2 \rangle \sim t^\gamma$, $\gamma < 1$, common in heterogeneous materials or systems with particle trapping.

Such anomalies cannot be handled by integer-order models. The FO derivatives' non-local characteristics encode memory effects. The FO temporal derivative $\frac{\partial^\varphi u}{\partial t^\varphi}$ encodes memory effects, while the FO spatial derivative $\frac{\partial^\sigma u}{\partial x^\sigma}$ encodes long-range interactions. The model (1) extends the classical advection equation to the FO framework with separate controls over temporal memory (via φ) and spatial non-locality (via σ). The homogeneous Neumann boundary condition (2) describes an insulated or impermeable boundary condition. The terminal condition (4) hints at a potential connection with a control or optimization problem, or merely specifies a final state for the system. Figure 1 illustrates the physicist's role in relating the FO model with physical phenomena.

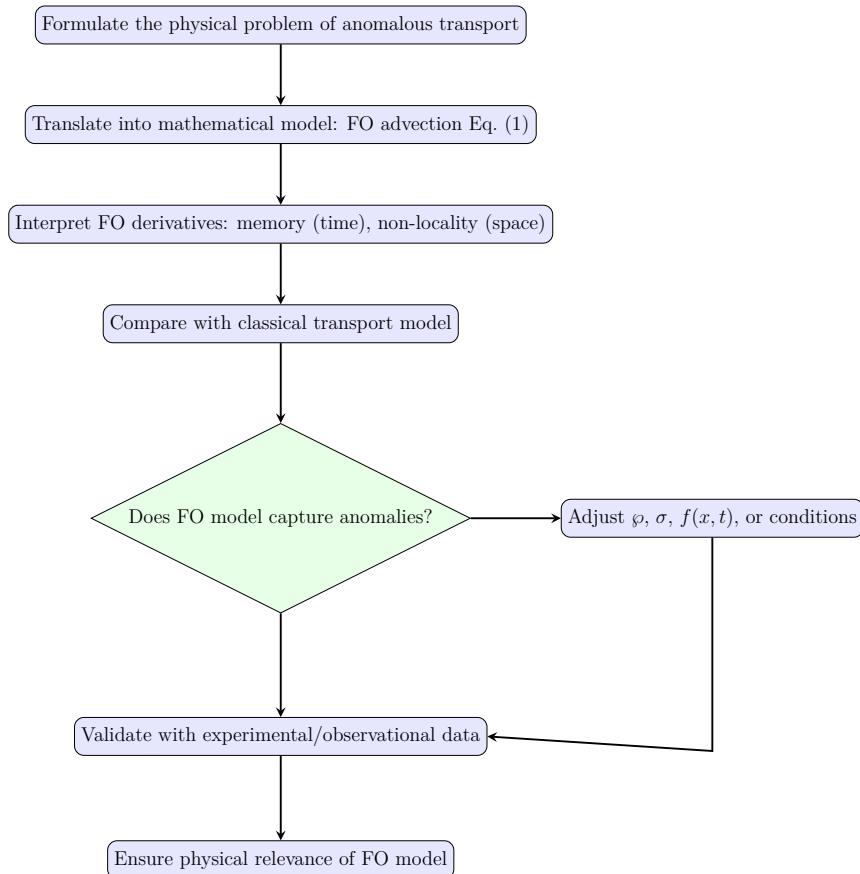


Figure 1. Compact flowchart of the physicist's role in FO advection modeling.

To numerically solve (1)–(4), we discretize the spatial domain $\Omega = [0, L]$ uniformly:

$$x_i = ih, \quad i = 0, 1, \dots, N, \quad t_j = jk, \quad j = 0, 1, \dots, M, \quad (8)$$

where h denotes the spatial mesh size and k represents the time discretization step. Let u_i^j and f_i^j denote the numerical approximations of $u(x_i, t_j)$ and $f(x_i, t_j)$. Using a FO Taylor series expansion, we approximate

the Caputo derivatives as follows:

$$\left(\frac{\partial^{\varrho} u}{\partial t^{\varrho}}\right)_i^j \approx \frac{u_i^{j+1} - u_i^j}{\Delta_1^*} \Gamma(\varrho + 1), \quad (9)$$

$$\left(\frac{\partial^{\sigma} u}{\partial x^{\sigma}}\right)_i^j \approx \frac{u_{i+1}^j - u_i^j}{\Delta_1} \Gamma(\sigma + 1), \quad (10)$$

where

$$\Delta_1^* = k^{\varrho} [(j+1)^{\varrho} - j^{\varrho}], \quad \Delta_1 = h^{\sigma} [(i+1)^{\sigma} - i^{\sigma}].$$

Substituting these into (1) yields the explicit FDS:

$$\begin{cases} u_i^{j+1} = (1-B)u_i^j + Bu_{i+1}^j + \frac{\Delta_1^*}{\Gamma(\varrho+1)} f_i^j, & i = 1, \dots, N-1; \quad j = 0, \dots, M-1, \\ u_0^j = u_1^j = 0, & j = 0, \dots, M, \\ u_N^j = u_{N+1}^j = 0, & j = 0, \dots, M, \\ u_i^0 = u_0(x_i), & i = 0, \dots, N, \\ u_i^M = u_1(x_i), & i = 0, \dots, N, \end{cases} \quad (11)$$

where $B = \frac{\Delta_1^* \Gamma(\sigma+1)}{\Delta_1 \Gamma(\varrho+1)}$.

3 Main Results

In this section, we will analyze the consistency, stability, and convergence of the explicit FDS as presented in the previous section. The stability and convergence analysis follow the standard procedure for the analysis of any numerical method. First, we analyze the local truncation error to show the consistency of the method. Next, we carry out the Fourier stability analysis, and lastly, we show the convergence of the method using the results obtained in the first two steps. The local truncation error is used to measure the accuracy with which the exact solution satisfies the FDS. From the approximations obtained using the FO Taylors series in Eqs (9)–(10), the local truncation error R_i^j of the FDS (11) at the point (x_i, t_j) is given by:

$$R_i^j = \left[\frac{\partial^{\varrho} u}{\partial t^{\varrho}}(x_i, t_j) - \frac{u_i^{j+1} - u_i^j}{\Delta_1^*} \Gamma(\varrho + 1) \right] + \left[\frac{\partial^{\sigma} u}{\partial x^{\sigma}}(x_i, t_j) - \frac{u_{i+1}^j - u_i^j}{\Delta_1} \Gamma(\sigma + 1) \right] \quad (12)$$

$$= R_1(x_i, t_{j+1}, t_j, 0) + R_2(x_{i+1}, x_i, t_j, 0), \quad i = 0, 1, \dots, N-1; \quad j = 0, 1, \dots, M-1. \quad (13)$$

The remainder terms R_1 and R_2 from the FO Taylor expansions are of order:

$$R_1(x_i, t_{j+1}, t_j, 0) = k^{2\varrho} \left[\frac{A_1 \sigma_1 j^{\varrho}}{\Gamma^2(\varrho+1)} - \frac{C_1 k^{\varrho} \sigma_1 j^{2\varrho}}{\Gamma(\varrho+1) \Gamma(2\varrho+1)} + \frac{(D_1 - B_1) j^{2\varrho}}{\Gamma(2\varrho+1)} \right], \quad (14)$$

$$R_2(x_{i+1}, x_i, t_j, 0) = h^{2\sigma} \left[\frac{A_2 \sigma_2 i^{\sigma}}{\Gamma^2(\sigma+1)} - \frac{C_2 h^{\sigma} \sigma_2 i^{2\sigma}}{\Gamma(\sigma+1) \Gamma(2\sigma+1)} + \frac{(D_2 - B_2) i^{2\sigma}}{\Gamma(2\sigma+1)} \right], \quad (15)$$

for $i = 0, 1, \dots, N-1; j = 0, 1, \dots, M-1$, where the constants are defined as:

$$\begin{aligned} A_1 &= \frac{\partial^{2\varrho} u}{\partial t^{2\varrho}}(x_i, 0), & B_1 &= \frac{\partial^{2\varrho} u}{\partial t^{2\varrho}}(x_i, \xi_1), & C_1 &= \frac{\partial^{2\varrho} u}{\partial t^{2\varrho}}(x_i, \xi_2), & D_1 &= \frac{\partial^{2\varrho} u}{\partial t^{2\varrho}}(x_i, \xi_3), \\ A_2 &= \frac{\partial^{2\sigma} u}{\partial x^{2\sigma}}(0, t_j), & B_2 &= \frac{\partial^{2\sigma} u}{\partial x^{2\sigma}}(\tau_1, t_j), & C_2 &= \frac{\partial^{2\sigma} u}{\partial x^{2\sigma}}(\tau_2, t_j), & D_2 &= \frac{\partial^{2\sigma} u}{\partial x^{2\sigma}}(\tau_3, t_j), \end{aligned}$$

and:

$$\begin{aligned} \sigma_1 &= (j+1)^{\varrho} - j^{\varrho}, & j &= 0, 1, \dots, M-1, \\ \sigma_2 &= (i+1)^{\sigma} - i^{\sigma}, & i &= 0, 1, \dots, N-1. \end{aligned}$$

Since $R_1 = \mathcal{O}(k^{2\varrho})$ and $R_2 = \mathcal{O}(h^{2\sigma})$, the overall local truncation error is $R_i^j = \mathcal{O}(k^{2\varrho} + h^{2\sigma})$, which tends to zero as $k, h \rightarrow 0$. Therefore, the numerical scheme is consistent.

Definition 3.1 ([45]). Consider a linear FDS with error mode represented by

$$\varepsilon_i^j = \xi^j e^{ikx_i}, \quad k \in \mathbb{R},$$

where ξ is the amplification factor. The scheme is said to be stable in the sense of Fourier if and only if

$$|\xi| \leq 1 \quad \text{for all wave numbers } k \text{ and for all time steps } j \geq 0. \quad (16)$$

Let $\rho_i^j = u_i^j - U_i^j$ be the difference between the numerical solution u_i^j and the exact solution of the difference equation U_i^j . The error ρ_i^j satisfies the homogeneous version of the numerical scheme:

$$\rho_i^{j+1} = \left(1 - \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)}\right) \rho_i^j + \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)} \rho_{i+1}^j, \quad i = 1, \dots, N; \quad j = 0, \dots, M-1, \quad (17)$$

$$\rho_i^0 = 0, \quad i = 0, \dots, N, \quad (18)$$

$$\rho_i^M = 0, \quad i = 0, \dots, N. \quad (19)$$

We assume a solution to the error equation of the form:

$$\rho_i^j = \xi_j e^{I\sigma ih}, \quad i = 0, 1, \dots, N; \quad j = 0, 1, \dots, M, \quad (20)$$

where $I = \sqrt{-1}$ and $\sigma = 2\pi m$ is the wave number. Substituting (20) into (17) yields:

$$\xi_{j+1} e^{I\sigma ih} = (1 - B) \xi_j e^{I\sigma ih} + B \xi_j e^{I\sigma(i+1)h}, \quad (21)$$

where $B = \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)}$. Dividing both sides by $e^{I\sigma ih}$, we obtain the recurrence relation for the amplification factor:

$$\xi_{j+1} = [1 - B + B e^{I\sigma h}] \xi_j, \quad j = 0, 1, \dots, M-1. \quad (22)$$

The term $G = 1 - B + B e^{I\sigma h}$ is the amplification factor. For stability, we require that the magnitude of this factor does not exceed unity for all wave numbers σ .

Theorem 3.1. The scheme (11) is stable if for all $i = 0, 1, \dots, N-1$ and $j = 0, 1, \dots, M-1$, the following condition holds:

$$\left| 1 - \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)} + \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)} e^{I\sigma h} \right| \leq 1, \quad \varrho, \sigma \in (0, 1). \quad (23)$$

Proof. Consider the error equation:

$$\rho_i^{j+1} = \left(1 - \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)}\right) \rho_i^j + \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)} \rho_{i+1}^j. \quad (24)$$

Assume the error can be expressed as a discrete Fourier series:

$$\rho_i^j = \sum_{m=-\infty}^{\infty} \xi_j(m) e^{I\vartheta_m ih}, \quad (25)$$

where $\vartheta_m = 2\pi m$. Substituting into (24):

$$\sum_m \xi_{j+1}(m) e^{I\vartheta_m ih} = (1 - B) \sum_m \xi_j(m) e^{I\vartheta_m ih} + B \sum_m \xi_j(m) e^{I\vartheta_m(i+1)h}, \quad (26)$$

where $B = \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)}$. For each mode m , we divide by $e^{I\vartheta_m ih}$:

$$\begin{aligned} \xi_{j+1}(m) &= (1 - B) \xi_j(m) + B \xi_j(m) e^{I\vartheta_m h} \\ &= G(i, j, m) \xi_j(m), \end{aligned} \quad (27)$$

with amplification factor:

$$G(i, j, m) = 1 - B + Be^{I\vartheta_m h}. \quad (28)$$

By the theorem's condition, $|G(i, j, m)| \leq 1$ for all i, j, m . Therefore,

$$|\xi_{j+1}(m)| = |G(i, j, m)| \cdot |\xi_j(m)| \leq |\xi_j(m)|. \quad (29)$$

By induction:

$$|\xi_j(m)| \leq |\xi_0(m)| \quad \text{for all } j. \quad (30)$$

The 2-norm of the error is:

$$\|\rho^j\|_2^2 = \sum_{m=-\infty}^{\infty} |\xi_j(m)|^2 \leq \sum_{m=-\infty}^{\infty} |\xi_0(m)|^2 = \|\rho^0\|_2^2. \quad (31)$$

Hence, $\|\rho^j\|_2 \leq \|\rho^0\|_2$ for all j , and the scheme is stable. $\square$

Definition 3.2 ([46]). Let $u(x, t)$ be the exact solution of the continuous problem and u_i^j the numerical solution obtained from a FDS. The scheme is said to be convergent in the sense of Fourier if, for any fixed final time $T = M\Delta t$, we have

$$\lim_{\substack{\Delta t \rightarrow 0 \\ \Delta x \rightarrow 0}} \max_{0 \leq j \leq M} \max_{0 \leq i \leq N} |u(x_i, t_j) - u_i^j| = 0. \quad (32)$$

Equivalently, in terms of Fourier analysis, convergence holds if the amplification factor ξ of the scheme satisfies

$$|\xi| \leq 1 \quad \text{and} \quad \lim_{\Delta t, \Delta x \rightarrow 0} \xi = e^{-\lambda(k)\Delta t}, \quad (33)$$

where $\lambda(k)$ is the exact Fourier symbol of the differential operator for all wave numbers k .

Let $e_i^j = u_i^j - u(x_i, t_j)$ be the error between the numerical solution and the exact solution of the continuous PDE. Subtracting the exact equation from the numerical scheme, we obtain the error Eq. (34):

$$e_i^{j+1} = \left(1 - \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)}\right) e_i^j + \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)} e_{i+1}^j + R_i^j, \quad i = 0, 1, \dots, N; \quad j = 0, 1, \dots, M. \quad (34)$$

Theorem 3.2. Assume the stability condition (23) holds and the local truncation error satisfies $\|R^j\|_\infty \leq C^*(k^{2\varrho} + h^{2\sigma})$ for all j . Then, the scheme (11) is convergent, and the error satisfies:

$$\|e^j\|_\infty \leq C^* M(k^{2\varrho} + h^{2\sigma}), \quad j = 0, 1, \dots, M, \quad \varrho, \sigma \in (0, 1). \quad (35)$$

Proof. Under the stability condition (23), which is equivalent to $0 \leq B \leq 1$ where

$$B = \frac{\Delta_1^* \Gamma(\sigma + 1)}{\Delta_1 \Gamma(\varrho + 1)}, \quad (36)$$

we analyze the error equation:

$$e_i^{j+1} = (1 - B)e_i^j + Be_{i+1}^j + R_i^j, \quad i = 0, 1, \dots, N; \quad j = 0, 1, \dots, M. \quad (37)$$

Taking absolute values and applying the triangle inequality:

$$|e_i^{j+1}| \leq (1 - B)|e_i^j| + B|e_{i+1}^j| + |R_i^j|. \quad (38)$$

Since $0 \leq B \leq 1$, we have $(1 - B) + B = 1$. Taking the maximum over i on both sides yields:

$$\|e^{j+1}\|_\infty \leq \|e^j\|_\infty + \|R^j\|_\infty. \quad (39)$$

Given that $\|e^0\|_\infty = 0$ (initial error is zero), we proceed by induction:

$$\|e^j\|_\infty \leq \sum_{m=0}^{j-1} \|R^m\|_\infty. \quad (40)$$

By assumption, $\|R^m\|_\infty \leq C^*(k^{2\varrho} + h^{2\sigma})$ for all m , so:

$$\|e^j\|_\infty \leq jC^*(k^{2\varrho} + h^{2\sigma}) \leq MC^*(k^{2\varrho} + h^{2\sigma}). \quad (41)$$

Thus, the scheme is convergent under the stated conditions. $\square$

The development of the proposed explicit FDS relies essentially on two complementary theoretical foundations:

- **FO Taylor Expansion:** This theory extends the classical Taylor series to FOs allowing the approximation of Caputo derivatives. It plays a crucial role in discretizing the time space fractional advection equation leading to a forward Euler type scheme that is simple, explicit and efficient. By capturing the nonlocal and memory properties inherent in FO calculus it ensures that the method accurately reflects the underlying physics of anomalous transport.
- **Fourier Stability Analysis:** This theory provides a rigorous framework for assessing the stability of numerical schemes. By analyzing amplification factors in the Fourier space it identifies the stability region and ensures that numerical errors remain bounded during time integration. Combined with consistency analysis it guarantees convergence of the method.

Together, these two theories facilitate the construction, validation, and theoretical foundation of the proposed scheme (Figure 2). The FO Taylor expansion develops the discretization, while Fourier stability analysis ensures its reliability, making the overall model both accurate and robust.

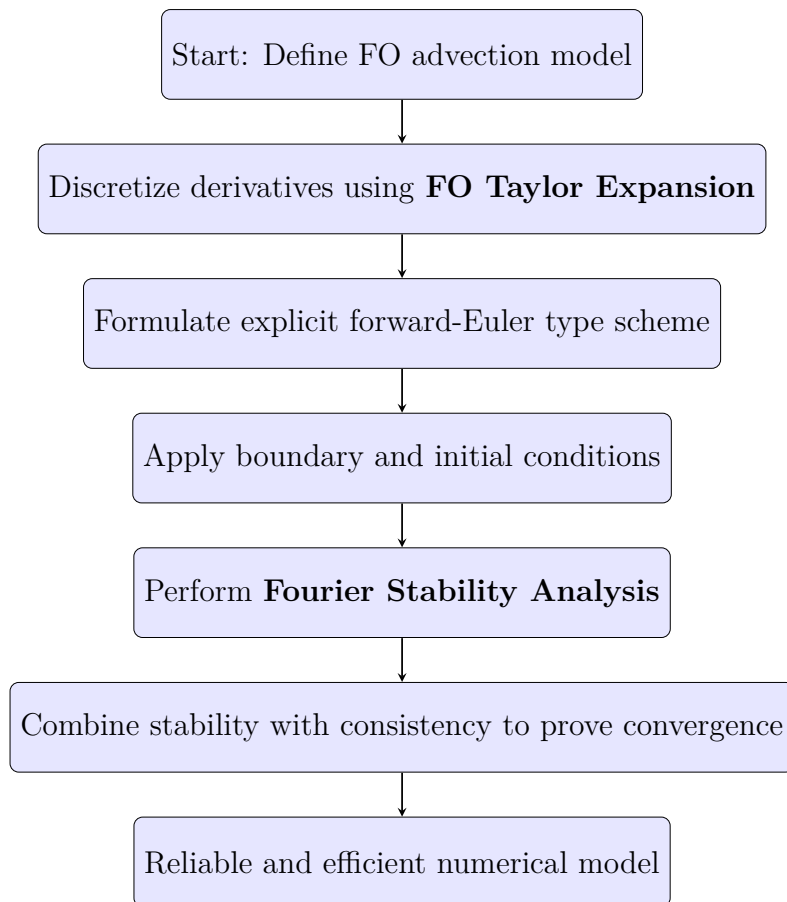


Figure 2. Contribution of the two theories in developing the proposed model.

4 Simulation Results

In this section, the explicit FDS developed in this study is implemented for solving two different test problems for linear time-space FO advection equations. The aim of these test problems is to validate the accuracy and efficiency of the numerical scheme for solving different scenarios of source terms, domains, and boundary conditions. Each of these test problems is intended to validate the applicability of the numerical scheme in solving FO PDEs with non-integer derivatives. The numerical method employed is the explicit FDS derived in Section 2, which is based on FO Taylor series expansions for approximating Caputo derivatives. The stability and convergence of this scheme have been rigorously analyzed in Section 3 (Theorems 3.1 and 3.2), ensuring its reliability under the specified conditions. The following test problems were solved using the proposed scheme (Table 1):

Table 1. Comparison of parameters for advection–diffusion equation.

Parameter	Eq. (1)	Eq. (2)
Time FO ρ	0.999	0.899
Space FO σ	0.999	0.979
Space domain Ω	[0, 5]	[0, 5]
Time domain $[0, T]$	[0, 5]	[0, 5]
Spatial discretization size N	100	100
Time discretization size M	100	100
Spatial mesh size h	0.05	0.05
Temporal mesh size k	0.05	0.05
Initial condition	$0.65 \sin(x)$	$0.85 \sin(x)$
Terminal condition	$0.75 \cos(x)$	$0.45 \cos(x)$
$f(x, t)$	$\frac{1}{1 + x^{\rho} t^{\sigma}}$	$\frac{1}{1 + 0.75 x^{\rho} t^{2\sigma}}$

The suggested explicit FDS was coded in MATLAB, and all test problems were solved numerically. The numerical solutions obtained show the effectiveness of the method in solving FO advection equations of different orders, source terms, and boundary conditions. The numerical solutions obtained satisfy the stability conditions presented in Theorem 3.1, while the convergence of the numerical solutions obtained satisfies the error bounds presented in Theorem 3.2. Figures 3 and 4 show the numerical solutions of Problems 1 and 2, respectively. The numerical solutions obtained show the expected physical behaviors of the solutions, including wave propagation and diffusion, as presented in the FO advection equations.

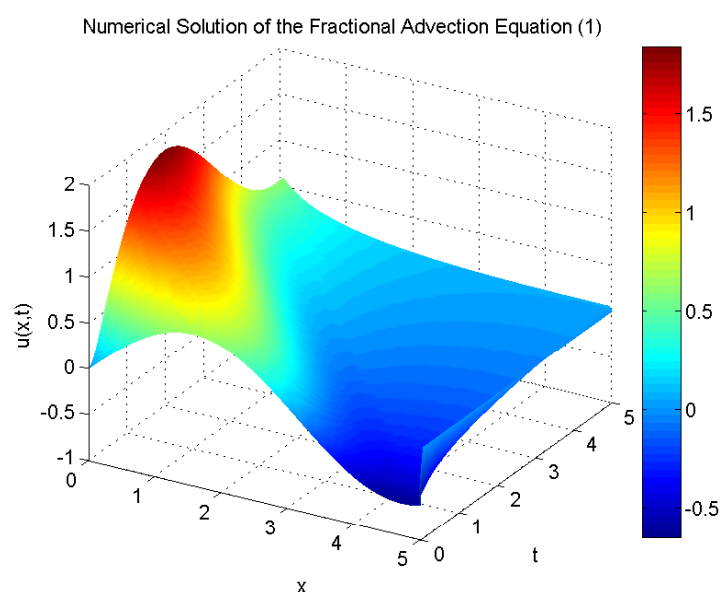


Figure 3. Numerical solution for Eq (1).

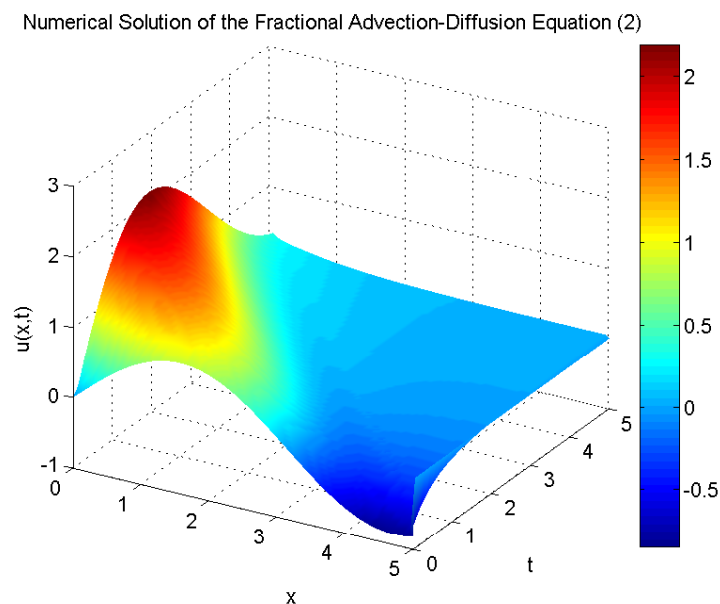


Figure 4. Numerical solution for Eq (2).

The results presented in Figures 3 through 4 are crucial to the validation of the theoretical results related to the stability and convergence properties of the proposed explicit FDS. The application of the proposed method to different test problems with different FO orders, source terms, and boundary conditions clearly validates the accuracy and reliability of the proposed method in capturing the non-local effects and memory characteristics of the FO operators. In addition, the accuracy of the obtained numerical solutions clearly validates the physical behavior and properties that can be observed in the results. Eq. (1), with $\varphi = 0.999$ and $\sigma = 0.999$, is close to the classical advection equation and has smoother behavior, which results in convergence. Eq. (2), with $\varphi = 0.899$ and $\sigma = 0.979$, has stronger memory effects and non-local behavior and is more difficult and challenging to solve using the FO operator; however, it is also the best test problem to evaluate the capabilities of the proposed method with respect to FO operators.

5 Conclusion

In the paper, we proposed a simple explicit FDS for solving the time-space FO advection equations with different FO orders in the time and space variables. The proposed method is based on the FO Taylor series expansion and has been rigorously analyzed with respect to its stability and convergence properties. Theoretical analysis has proved that the proposed method is stable under certain conditions and has convergence properties with an error bound depending on the discretization parameters. To confirm the accuracy of the proposed method, several numerical tests were carried out on different test problems with different FO orders, source terms, and boundary conditions. The accuracy and efficiency of the proposed method were clearly presented in the numerical tests, as illustrated in Figures 3 and 4. The results show that the proposed method can efficiently handle the non-local memory effects in the FO derivative and can be used to solve the FO PDEs. In the future more research can focus on the multi dimensional space and nonlinear FO equations as well as the adaptive methods for accuracy and efficiency.

6 Open Problem

Several directions remain open. Extending the proposed explicit FDS to multidimensional and nonlinear time-space FO advection equations requires new consistency, stability, and convergence analyses. Sharp error estimates for nonsmooth or weakly singular solutions, adaptive time-stepping and spatial refinement strategies, and high-order explicit schemes with enlarged stability regions are still needed. Another open question is the derivation of explicit stability constraints for variable-order and distributed-order FO advection models, as well as the coupling of such schemes with the control or optimization problems suggested by the terminal condition.

Appendix

Figure 5 summarizes the complete workflow of the proposed explicit FDS.

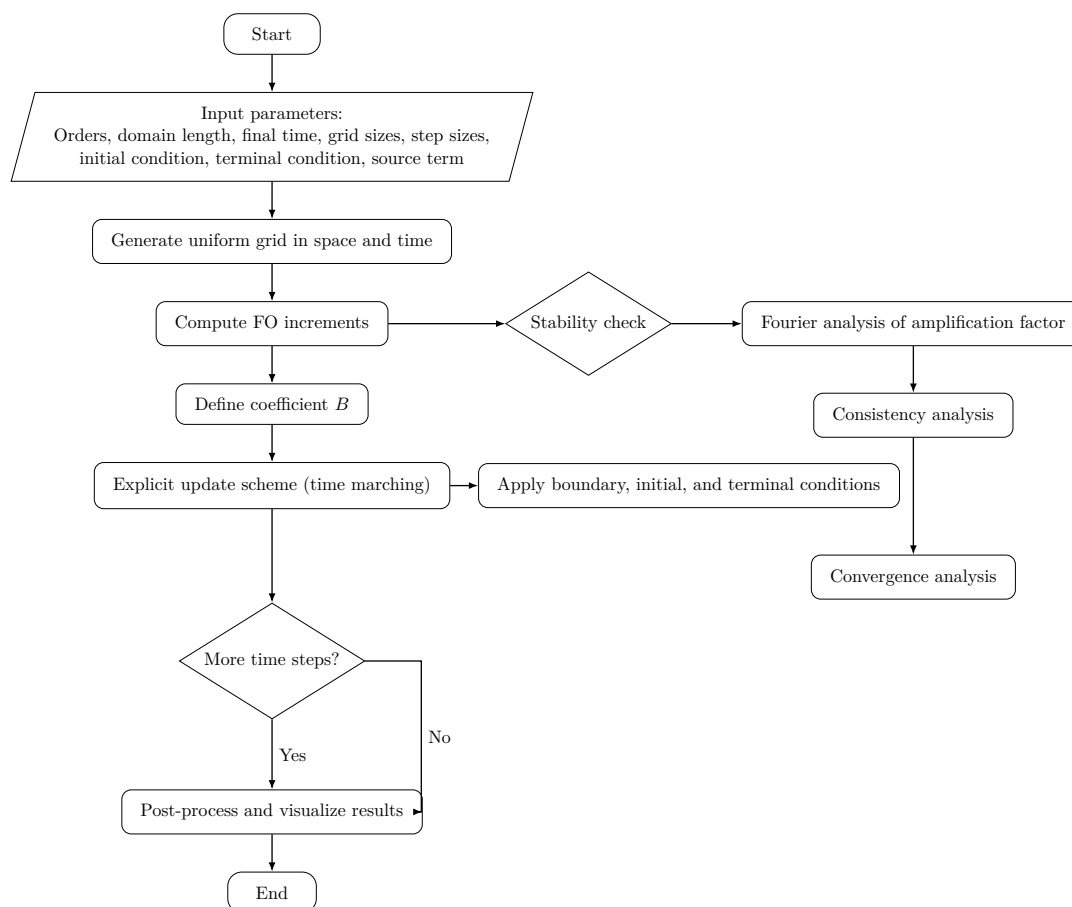


Figure 5. Workflow of the explicit FDS via a FO Taylor expansion.

Declarations

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Conflicts of interest. The authors declare that they have no competing interests.

Ethics approval. Not applicable.

Informed consent. Not applicable.

Data availability. The data supporting the findings of this study are available from the corresponding author upon reasonable request.

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Use of artificial intelligence. No AI tools were used beyond language editing.

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