RESEARCH ARTICLE

Quantitative Symmetry Breaking of Hitting Times at Cut Vertices

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ABSTRACT

For a simple random walk on a finite connected simple graph, we study how cut vertices create asymmetry of hitting times. Let c be a cut vertex with branches $G_1, \ldots, G_s$, and write $m_i = |E(G_i)|$ and $M = |E(G)|$. Using the cut-vertex decomposition of hitting times, we derive, for every $u \in V(G_i) \setminus \{c\}$, the local-to-global ratio formula

$$\frac{H_G(c,u)}{H_G(u,c)} = \frac{M - m_i}{m_i} + \frac{M}{m_i}\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)}.$$

In particular,

$$\frac{H_G(c,u)}{H_G(u,c)} > \frac{M - m_i}{m_i}.$$

Consequently, at least $s - 1$ branches G_i satisfy

$$H_G(c,u) > H_G(u,c)$$

for every $u \in V(G_i) \setminus \{c\}$, and on a branch having the minimum number of edges the ratio is greater than $s - 1$. We show that this constant is asymptotically best possible. These results imply that no finite connected simple graph with a cut vertex is highly symmetric and that K_2 is the only finite connected simple highly symmetric graph with a cut edge. This answers two questions posed by Palacios and Markowsky. We also combine the general reversibility identity for hitting times with the cut-vertex formulas to express the asymmetry between vertices lying in different branches explicitly in terms of local branch data.

Keywords: hitting time; cut vertex; cut edge; effective resistance; highly symmetric graph; vertex identification.

Mathematics Subject Classification: Primary 05C81; Secondary 05C40, 60J10.

1 Introduction

Let G be a finite connected simple graph, and consider a simple random walk on G. For vertices $u, v \in V(G)$, let $H_G(u,v)$ denote the expected number of steps required for a random walk starting at u to visit v for the first time. In general, hitting times are not symmetric:

$$H_G(u,v) \neq H_G(v,u).$$

On the other hand, if $R_G(u,v)$ denotes the effective resistance between u and v when every edge of G is regarded as a unit resistor, then the commute-time identity gives

$$H_G(u,v) + H_G(v,u) = 2|E(G)|R_G(u,v)$$

[1, 2].

A graph G is called highly symmetric if

$$H_G(u, v) = H_G(v, u)$$

for every pair of vertices $u, v \in V(G)$. Highly symmetric graphs were introduced and studied by Devroye and Sbihi [3], and later considered from several points of view [4–7]. By the commute-time identity, G is highly symmetric if and only if

$$H_G(u, v) = |E(G)|R_G(u, v)$$

for all $u, v \in V(G)$.

The behavior of hitting times in graphs with cut vertices has also been studied extensively. Kirkland and Neumann [8], Chen and Zhang [9], Palacios, Gómez and Del Río [10], and Del Río and Palacios [11] developed decomposition formulas and related methods for expressing hitting times on graphs with cut vertices in terms of quantities associated with smaller subgraphs. The cut-vertex decomposition formulas used in this paper are available in the previous literature. Our contribution is to derive new quantitative consequences from these formulas. In particular, we obtain an exact local-to-global formula for the hitting-time ratio and universal lower bounds depending only on the numbers of edges in the branches and in the whole graph.

These results describe both the direction and magnitude of the hitting-time asymmetry forced by a cut vertex.

Palacios and Markowsky [6] proved that a highly symmetric graph cannot have a resistance-regular cut vertex and asked whether a highly symmetric graph can have a cut vertex at all. They also asked whether there exists a highly symmetric graph other than K_2 having a cut edge.

We answer both questions in the negative and provide a quantitative description of the symmetry breaking produced by a cut vertex.

Let c be a cut vertex of G, and let

$$C_1, \ldots, C_s, \qquad s \geq 2,$$

be the connected components of $G - c$. For each i, let

$$G_i = G[V(C_i) \cup \{c\}]$$

be the corresponding branch at c, and set

$$m_i = |E(G_i)|, \qquad M = |E(G)| = \sum_{i=1}^{s} m_i.$$

For every $u \in V(G_i) \setminus \{c\}$, we first record the cut-vertex identities

$$H_G(u, c) = H_{G_i}(u, c)$$

and

$$H_G(c, u) = H_{G_i}(c, u) + 2(M - m_i)R_{G_i}(c, u).$$

Thus, attaching the other branches at c does not change the hitting time from u to c, whereas the hitting time in the opposite direction increases by an explicit resistance term.

These identities yield an exact relation between the hitting-time ratio in G and the corresponding ratio within the branch G_i. For every $u \in V(G_i) \setminus \{c\}$,

$$\frac{H_G(c, u)}{H_G(u, c)} = \frac{M - m_i}{m_i} + \frac{M}{m_i} \frac{H_{G_i}(c, u)}{H_{G_i}(u, c)}.$$

Thus the global hitting-time ratio is completely determined by the numbers of edges in the branch and in the whole graph together with the corresponding local ratio inside G_i. Since

$$\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} > 0,$$

we obtain the universal bound

$$\frac{H_G(c,u)}{H_G(u,c)} > \frac{M-m_i}{m_i}.$$

The right-hand side depends only on the numbers of edges in the branch and in the whole graph, and is independent of the internal structure of the branch.

This inequality gives several consequences. If

$$m_i \leq \frac{M}{2},$$

then

$$H_G(c,u) > H_G(u,c)$$

for every $u \in V(G_i) \setminus \{c\}$. Since at most one branch can contain more than half of the edges of G, the same strict inequality holds on at least $s-1$ entire branches. In particular, if G_i is a branch having the minimum number of edges, then

$$m_i \leq \frac{M}{s},$$

and hence

$$\frac{H_G(c,u)}{H_G(u,c)} > s-1$$

for every $u \in V(G_i) \setminus \{c\}$. Thus a cut vertex with many branches necessarily produces a quantitatively large directional bias in at least one branch.

The constant $s-1$ in this estimate is asymptotically best possible if only the number of branches is prescribed. Indeed, we construct a family of graphs with s branches for which

$$\frac{H_G(c,u)}{H_G(u,c)} \longrightarrow s-1$$

as the number of edges in each branch tends to infinity. Thus the lower bound cannot be replaced by any larger constant depending only on s.

Define the hitting-time asymmetry by

$$A_G(x,y) = H_G(x,y) - H_G(y,x).$$

By the general reversibility identity for hitting times [10, Theorem 11], for arbitrary vertices $x, y, z \in V(G)$,

$$A_G(x,y) = A_G(z,y) - A_G(z,x).$$

This potential-difference identity is not specific to cut vertices. In Section 5, we combine it with the cut-vertex formulas obtained above to express the asymmetry between vertices lying in different branches explicitly in terms of local branch data.

As an immediate qualitative consequence of the quantitative estimates, we prove that no finite connected simple graph with a cut vertex is highly symmetric. Therefore every finite connected highly symmetric graph with at least three vertices is 2-connected. It also follows that K_2 is the only finite connected simple highly symmetric graph having a cut edge. These results settle the two questions of Palacios and Markowsky mentioned above.

We also obtain sharper formulas when a branch is itself highly symmetric. If G_i is highly symmetric, then for every $u \in V(G_i) \setminus \{c\}$,

$$H_G(u,c) = m_i R_{G_i}(c,u)$$

and

$$H_G(c,u) = (2M - m_i)R_{G_i}(c,u).$$

In particular,

$$\frac{H_G(c,u)}{H_G(u,c)} = \frac{2M - m_i}{m_i},$$

which is independent of the choice of u in the branch. If s copies of the same highly symmetric graph are identified at one common vertex, this ratio becomes

$$2s - 1,$$

recovering the corresponding relation for conjoined highly symmetric graphs studied by Palacios and Markowsky [7].

Finally, the results apply directly to one-vertex identifications. If two or more nontrivial connected graphs are identified at one chosen vertex, the common vertex is a cut vertex, and therefore the resulting graph cannot be highly symmetric. The quantitative formulas obtained here give, in addition, explicit information on the direction and size of the hitting-time asymmetry created by the identification.

The paper is organized as follows. Section 2 recalls the basic definitions and the commute-time identity. In Section 3, we record the relevant cut-vertex decomposition formulas, establish an exact local-to-global formula for the hitting-time ratio, and obtain quantitative asymmetry bounds. Section 4 applies these bounds to highly symmetric graphs and proves that the bound depending only on the number of branches is asymptotically sharp. Section 5 combines the general reversibility identity for hitting times with our cut-vertex formulas to obtain explicit cross-branch representations, and then studies highly symmetric branches and applications to one-vertex identifications. Section 6 summarizes our main results. Finally, Section 7 presents an open problem concerning the structure of graphs with symmetric hitting times.

2 Preliminaries

Throughout this paper, let $G = (V(G), E(G))$ be a finite connected simple graph, and write

$$m(G) = |E(G)|.$$

We consider the simple random walk on G, which moves at each step from its current vertex to one of its neighbors chosen uniformly at random.

For vertices $u, v \in V(G)$, let $H_G(u,v)$ denote the expected number of steps required for a simple random walk starting at u to visit v for the first time. We also write $R_G(u,v)$ for the effective resistance between u and v when every edge of G is regarded as a unit resistor.

The basic relation between these two quantities is the commute-time identity [1,2]:

$$H_G(u,v) + H_G(v,u) = 2m(G)R_G(u,v).$$

We shall measure the directional asymmetry of hitting times by

$$A_G(u,v) = H_G(u,v) - H_G(v,u).$$

Then

$$A_G(v,u) = -A_G(u,v),$$

and the commute-time identity gives

$$H_G(u,v) = m(G)R_G(u,v) + \frac{1}{2}A_G(u,v).$$

Thus the hitting time between two vertices is determined by its symmetric resistance contribution together with the asymmetry term $A_G(u,v)$.

We shall also use the following general reversibility identity for hitting times.

Proposition 2.1 ([10], Theorem 11)**.** *For arbitrary vertices $u, v, w \in V(G)$,*

$$H_G(u,v) + H_G(v,w) + H_G(w,u) = H_G(u,w) + H_G(w,v) + H_G(v,u).$$

Consequently,

$$A_G(u,v) = A_G(w,v) - A_G(w,u).$$

Proof. The first identity is the reversibility of the sum of hitting times around a triangle proved in [10, Theorem 11]. Rearranging its terms gives

$$\begin{aligned} A_G(u,v) &= H_G(u,v) - H_G(v,u) \\ &= \big(H_G(w,v) - H_G(v,w)\big) - \big(H_G(w,u) - H_G(u,w)\big) \\ &= A_G(w,v) - A_G(w,u). \end{aligned}$$

□

A graph G is called highly symmetric if

$$H_G(u,v) = H_G(v,u)$$

for all $u, v \in V(G)$. Equivalently,

$$A_G(u,v) = 0$$

for every pair of vertices u, v. By the commute-time identity, this is also equivalent to

$$H_G(u,v) = m(G)R_G(u,v)$$

for all $u, v \in V(G)$.

A vertex $c \in V(G)$ is called a cut vertex if $G - c$ is disconnected. If

$$C_1, \ldots, C_s, \qquad s \geq 2,$$

are the connected components of $G - c$, then for each i we define

$$G_i = G[V(C_i) \cup \{c\}]$$

and call G_i the ith branch of G at c. We write

$$m_i = |E(G_i)|, \qquad M = |E(G)|.$$

Since the branches have pairwise disjoint edge sets,

$$M = \sum_{i=1}^{s} m_i.$$

An edge $e \in E(G)$ is called a cut edge if $G - e$ is disconnected.

The following elementary observation will be used repeatedly. If

$$u \in V(G_i) \setminus \{c\},$$

then every path from u to a vertex outside G_i must pass through c. Consequently, a random walk starting at u cannot leave the branch before first reaching c. This simple separation property is the basis of the hitting-time decompositions in the next section.

3 Hitting Times at a Cut Vertex

Let c be a cut vertex of G, and retain the notation introduced in Section 2. Thus

$$G_1, \ldots, G_s, \qquad s \geq 2,$$

are the branches of G at c, and

$$m_i = |E(G_i)|, \qquad M = |E(G)| = \sum_{i=1}^{s} m_i.$$

We first record the locality of hitting times toward the cut vertex and of effective resistance.

Lemma 3.1. *Let*

$$u \in V(G_i) \setminus \{c\}.$$

Then

$$H_G(u, c) = H_{G_i}(u, c)$$

and

$$R_G(u, c) = R_{G_i}(u, c).$$

Proof. Since C_i is a connected component of $G - c$, a random walk starting at u cannot leave C_i before first reaching c. Moreover, for every $x \in C_i$, the degree of x in G is the same as its degree in G_i. Hence the transition probabilities before the first visit to c are identical in G and in G_i. Therefore,

$$H_G(u, c) = H_{G_i}(u, c).$$

For the effective resistance, every branch other than G_i is attached to G_i only at the vertex c. If a potential difference is applied between u and c, the potential is constant on each of the other branches, and hence no current flows through them. Consequently, attaching these branches does not change the effective resistance between u and c. Thus

$$R_G(u, c) = R_{G_i}(u, c).$$

□

Cutpoint decomposition formulas of this type appear in [10, 11]. For completeness, and because this formula is the starting point of our quantitative estimates, we record the following specialization and give a short proof.

Theorem 3.2. *Let*

$$u \in V(G_i) \setminus \{c\}.$$

Then

$$H_G(c, u) = H_{G_i}(c, u) + 2(M - m_i)R_{G_i}(c, u).$$

Proof. Applying the commute-time identity to G, we obtain

$$H_G(c, u) + H_G(u, c) = 2MR_G(c, u).$$

By Lemma 3.1,

$$H_G(u, c) = H_{G_i}(u, c)$$

and

$$R_G(c, u) = R_{G_i}(c, u).$$

Hence

$$H_G(c, u) = 2MR_{G_i}(c, u) - H_{G_i}(u, c).$$

On the other hand, applying the commute-time identity to G_i gives

$$H_{G_i}(c,u) + H_{G_i}(u,c) = 2m_i R_{G_i}(c,u),$$

and therefore

$$H_{G_i}(u,c) = 2m_i R_{G_i}(c,u) - H_{G_i}(c,u).$$

Substitution yields

$$\begin{aligned} H_G(c,u) &= 2MR_{G_i}(c,u) - \big(2m_i R_{G_i}(c,u) - H_{G_i}(c,u)\big) \\ &= H_{G_i}(c,u) + 2(M-m_i)R_{G_i}(c,u). \end{aligned}$$

□

Theorem 3.2 shows a fundamental directional difference. The branches outside G_i have no effect on the hitting time from u to c, but they increase the hitting time from c to u by the explicit quantity

$$2(M-m_i)R_{G_i}(c,u).$$

This already suggests that a cut vertex should create an unavoidable asymmetry.

The following consequence gives two useful forms of this asymmetry.

Corollary 3.3. *Let*

$$u \in V(G_i) \setminus \{c\}.$$

Then

$$A_G(c,u) = A_{G_i}(c,u) + 2(M-m_i)R_{G_i}(c,u).$$

Equivalently,

$$A_G(c,u) = 2H_{G_i}(c,u) + 2(M-2m_i)R_{G_i}(c,u).$$

Proof. By definition,

$$A_G(c,u) = H_G(c,u) - H_G(u,c).$$

Using Theorem 3.2 and Lemma 3.1, we obtain

$$\begin{aligned} A_G(c,u) &= H_{G_i}(c,u) + 2(M-m_i)R_{G_i}(c,u) - H_{G_i}(u,c) \\ &= A_{G_i}(c,u) + 2(M-m_i)R_{G_i}(c,u). \end{aligned}$$

Furthermore, the commute-time identity in G_i gives

$$H_{G_i}(u,c) = 2m_i R_{G_i}(c,u) - H_{G_i}(c,u).$$

Hence

$$\begin{aligned} A_G(c,u) &= H_{G_i}(c,u) - \big(2m_i R_{G_i}(c,u) - H_{G_i}(c,u)\big) + 2(M-m_i)R_{G_i}(c,u) \\ &= 2H_{G_i}(c,u) + 2(M-2m_i)R_{G_i}(c,u). \end{aligned}$$

□

We now derive an exact formula relating the hitting-time ratio in G to the corresponding ratio within the branch G_i. As an immediate consequence, we obtain a universal lower bound depending only on the number of edges in the branch and in the whole graph.

Theorem 3.4. *For every*

$$u \in V(G_i) \setminus \{c\},$$

we have

$$\frac{H_G(c,u)}{H_G(u,c)} = \frac{M-m_i}{m_i} + \frac{M}{m_i}\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)}.$$

In particular,

$$\frac{H_G(c,u)}{H_G(u,c)} > \frac{M-m_i}{m_i}.$$

Proof. By Lemma 3.1 and Theorem 3.2,

$$\frac{H_G(c,u)}{H_G(u,c)} = \frac{H_{G_i}(c,u) + 2(M-m_i)R_{G_i}(c,u)}{H_{G_i}(u,c)}.$$

By the commute-time identity in G_i,

$$2m_i R_{G_i}(c,u) = H_{G_i}(c,u) + H_{G_i}(u,c).$$

Hence

$$\begin{aligned}\frac{H_G(c,u)}{H_G(u,c)} &= \frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} + \frac{M-m_i}{m_i}\frac{H_{G_i}(c,u)+H_{G_i}(u,c)}{H_{G_i}(u,c)} \\ &= \frac{M-m_i}{m_i} + \frac{M}{m_i}\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)}.\end{aligned}$$

Since G_i is connected and $u \neq c$,

$$H_{G_i}(c,u) > 0 \qquad \text{and} \qquad H_{G_i}(u,c) > 0.$$

Therefore,

$$\frac{H_G(c,u)}{H_G(u,c)} > \frac{M-m_i}{m_i}.$$

□

The exact identity in Theorem 3.4 depends on the local hitting-time ratio within the branch G_i. However, its universal lower bound

$$\frac{H_G(c,u)}{H_G(u,c)} > \frac{M-m_i}{m_i}$$

depends only on m_i and M. The exact identity also gives a complete criterion for the direction of the asymmetry.

Corollary 3.5. *For every*

$$u \in V(G_i) \setminus \{c\},$$

we have

$$H_G(c,u) > H_G(u,c)$$

if and only if

$$\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} > \frac{2m_i - M}{M}.$$

Proof. By Theorem 3.4,

$$\frac{H_G(c,u)}{H_G(u,c)} = \frac{M-m_i}{m_i} + \frac{M}{m_i}\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)}.$$

Therefore,

$$H_G(c,u) > H_G(u,c)$$

is equivalent to

$$\frac{M-m_i}{m_i} + \frac{M}{m_i}\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} > 1.$$

Multiplying by m_i and rearranging gives

$$M\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} > 2m_i - M,$$

which is equivalent to

$$\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} > \frac{2m_i - M}{M}.$$

□

Corollary 3.6. *If*

$$m_i \leq \frac{M}{2},$$

then

$$H_G(c,u) > H_G(u,c)$$

for every

$$u \in V(G_i) \setminus \{c\}.$$

Proof. If

$$m_i \leq \frac{M}{2},$$

then

$$\frac{2m_i - M}{M} \leq 0.$$

On the other hand,

$$\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} > 0.$$

Hence

$$\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} > \frac{2m_i - M}{M},$$

and the result follows from Corollary 3.5.

□

Since at most one branch can satisfy $m_i > M/2$, the preceding corollary has a useful global consequence.

Corollary 3.7. *Let c have s branches. Then at least $s-1$ branches G_i satisfy*

$$H_G(c,u) > H_G(u,c)$$

for every

$$u \in V(G_i) \setminus \{c\}.$$

Proof. There can be at most one index i such that

$$m_i > \frac{M}{2},$$

because two such branches would already contain more than M edges. Thus at least $s-1$ branches satisfy

$$m_i \leq \frac{M}{2}.$$

The conclusion follows from Corollary 3.6. □

Remark 3.1. The preceding argument also gives a universal additive form of the symmetry breaking. Indeed, if

$$m_i \leq \frac{M}{2},$$

then Corollary 3.3 gives

$$\begin{aligned} A_G(c,u) &= 2H_{G_i}(c,u) + 2(M-2m_i)R_{G_i}(c,u) \\ &\geq 2H_{G_i}(c,u) \geq 2 \end{aligned}$$

for every $u \in V(G_i) \setminus \{c\}$. Here the last inequality follows from $u \neq c$, since at least one step is required to hit u starting from c. Consequently, on at least $s-1$ branches,

$$H_G(c,u) - H_G(u,c) \geq 2$$

for every vertex u in the branch other than c. Thus the symmetry breaking has not only a fixed direction on these branches, but also a universal additive gap.

We also obtain a bound depending only on the number of branches.

Corollary 3.8. *Let G_i be a branch having the minimum number of edges. Then for every*

$$u \in V(G_i) \setminus \{c\},$$

$$\frac{H_G(c,u)}{H_G(u,c)} > s-1.$$

Proof. Since G_i has the minimum number of edges,

$$m_i \leq \frac{1}{s}\sum_{j=1}^{s} m_j = \frac{M}{s}.$$

Therefore,

$$\frac{M-m_i}{m_i} = \frac{M}{m_i} - 1 \geq s-1.$$

The result now follows from Theorem 3.4.

□

Thus a cut vertex does more than merely destroy the possible symmetry of hitting times. If the cut vertex separates the graph into many branches, then at least one entire branch exhibits a directional bias whose multiplicative size is bounded below solely in terms of the number of branches. In the next section, we show that this phenomenon immediately rules out highly symmetric graphs with cut vertices, and we prove that the constant $s-1$ in Corollary 3.8 is asymptotically best possible.

4 Symmetry Breaking and Sharpness

We now apply the quantitative estimates obtained in the previous section to the symmetry of hitting times. We first obtain the main qualitative consequence.

Theorem 4.1. *A finite connected simple graph with a cut vertex is not highly symmetric.*

Proof. Let c be a cut vertex of G, and suppose that c has $s \geq 2$ branches. Choose a branch G_i having the minimum number of edges. By Corollary 3.8, for every

$$u \in V(G_i) \setminus \{c\},$$

we have

$$\frac{H_G(c,u)}{H_G(u,c)} > s-1 \geq 1.$$

Hence

$$H_G(c,u) > H_G(u,c),$$

and therefore G is not highly symmetric.

□

The preceding quantitative results give more than the nonexistence statement in Theorem 4.1. By Corollary 3.7, if c has s branches, then at least $s-1$ entire branches satisfy

$$H_G(c,u) > H_G(u,c)$$

for every vertex u in the branch other than c. Thus the symmetry breaking caused by a cut vertex is not localized at a single pair of vertices.

Theorem 4.1 immediately answers the first question of Palacios and Markowsky mentioned in the Introduction.

Corollary 4.2. *Every finite connected highly symmetric graph with at least three vertices is* 2*-connected.*

Proof. Let G be a finite connected highly symmetric graph with at least three vertices. If G were not 2-connected, then G would have a cut vertex. This contradicts Theorem 4.1.

□

We also obtain the corresponding result for cut edges.

Corollary 4.3. *The only finite connected simple highly symmetric graph having a cut edge is* K_2*.*

Proof. Let G be a finite connected simple highly symmetric graph having a cut edge $e = xy$. Suppose that

$$|V(G)| \geq 3.$$

The graph $G - e$ has two connected components, one containing x and the other containing y. Since G has at least three vertices, at least one of these components contains more than one vertex. The endpoint of e belonging to that component is then a cut vertex of G.

By Theorem 4.1, this is impossible. Therefore

$$|V(G)| = 2,$$

and since G is connected and simple,

$$G = K_2.$$

Conversely, K_2 is clearly highly symmetric and its unique edge is a cut edge.

□

We next show that the lower bound in Corollary 3.8 is asymptotically best possible if only the number of branches is prescribed.

Proposition 4.4. *Fix an integer* $s \geq 2$*. For every* $\varepsilon > 0$*, there exists a finite connected simple graph* G *with a cut vertex* c *having exactly* s *branches and a vertex* u *lying in a branch with the minimum number of edges such that*

$$s - 1 < \frac{H_G(c,u)}{H_G(u,c)} < s - 1 + \varepsilon.$$

Consequently, the constant $s - 1$ *in Corollary 3.8 cannot be replaced by any larger constant depending only on* s*.*

Proof. For an integer $q \geq 2$, let F_q be the graph obtained from the complete graph K_q by adjoining one new vertex c and one new edge cu, where u is a fixed vertex of K_q. Set

$$m = |E(F_q)| = \binom{q}{2} + 1.$$

The edge cu is a cut edge of F_q, and hence

$$R_{F_q}(c,u) = 1.$$

Since c has degree one in F_q,

$$H_{F_q}(c,u) = 1.$$

By the commute-time identity,

$$H_{F_q}(c,u) + H_{F_q}(u,c) = 2mR_{F_q}(c,u) = 2m,$$

and therefore

$$H_{F_q}(u,c) = 2m - 1.$$

Now take s copies of F_q and identify their copies of the pendant vertex c into one common vertex, again denoted by c. Let the resulting graph be $G_{s,q}$. Then c is a cut vertex with exactly s branches, each having m edges, and

$$|E(G_{s,q})| = sm.$$

In particular, every branch has the minimum number of edges among the s branches.

Choose the vertex u adjacent to c in one of the branches. By Lemma 3.1,

$$H_{G_{s,q}}(u,c) = H_{F_q}(u,c) = 2m-1.$$

By Theorem 3.2,

$$\begin{aligned} H_{G_{s,q}}(c,u) &= H_{F_q}(c,u) + 2(sm-m)R_{F_q}(c,u) \\ &= 1 + 2(s-1)m. \end{aligned}$$

Hence

$$\frac{H_{G_{s,q}}(c,u)}{H_{G_{s,q}}(u,c)} = \frac{2(s-1)m+1}{2m-1}.$$

Since

$$2(s-1)m+1 = (s-1)(2m-1)+s,$$

we obtain the exact expression

$$\frac{H_{G_{s,q}}(c,u)}{H_{G_{s,q}}(u,c)} = s-1+\frac{s}{2m-1}.$$

As $q \to \infty$, we have $m \to \infty$, and therefore

$$\frac{H_{G_{s,q}}(c,u)}{H_{G_{s,q}}(u,c)} \longrightarrow s-1.$$

Thus, for every $\varepsilon > 0$, choosing q sufficiently large gives

$$s-1 < \frac{H_{G_{s,q}}(c,u)}{H_{G_{s,q}}(u,c)} < s-1+\varepsilon.$$

This proves the result.

□

Proposition 4.4 shows that the estimate

$$\frac{H_G(c,u)}{H_G(u,c)} > s-1$$

has the best possible universal constant when no information other than the number of branches is available. Thus the quantitative symmetry breaking obtained in Section 3 is sharp in this natural sense.

5 Cross-Branch Asymmetry and Highly Symmetric Branches

We now consider hitting times between vertices belonging to different branches of a cut vertex. The separation property of the cut vertex gives an exact decomposition of the corresponding hitting times.

For the asymmetry term, Proposition 2.1 gives a general potential-difference identity valid for arbitrary vertices. The role of the cut-vertex structure in this section is to combine this general identity with the formulas obtained in Section 3, thereby expressing the cross-branch asymmetry explicitly in terms of local branch quantities.

Let c be a cut vertex of G, with branches

$$G_1, \ldots, G_s,$$

and retain the notation

$$m_i = |E(G_i)|, \qquad M = |E(G)|.$$

Theorem 5.1. *Let*

$$u \in V(G_i) \setminus \{c\}, \qquad v \in V(G_j) \setminus \{c\}, \qquad i \neq j.$$

Then

$$H_G(u,v) = H_{G_i}(u,c) + H_{G_j}(c,v) + 2(M - m_j)R_{G_j}(c,v).$$

Similarly,

$$H_G(v,u) = H_{G_j}(v,c) + H_{G_i}(c,u) + 2(M - m_i)R_{G_i}(c,u).$$

Proof. Every path from u to v must pass through c. Hence, by the strong Markov property applied at the first hitting time of c,

$$H_G(u,v) = H_G(u,c) + H_G(c,v).$$

By Lemma 3.1,

$$H_G(u,c) = H_{G_i}(u,c),$$

while Theorem 3.2 gives

$$H_G(c,v) = H_{G_j}(c,v) + 2(M - m_j)R_{G_j}(c,v).$$

Therefore,

$$H_G(u,v) = H_{G_i}(u,c) + H_{G_j}(c,v) + 2(M - m_j)R_{G_j}(c,v).$$

Interchanging u and v gives

$$H_G(v,u) = H_{G_j}(v,c) + H_{G_i}(c,u) + 2(M - m_i)R_{G_i}(c,u).$$

□

We now turn to the asymmetry between vertices lying in different branches.

By Proposition 2.1, for arbitrary vertices $x, y, w \in V(G)$,

$$A_G(x,y) = A_G(w,y) - A_G(w,x).$$

The general identity does not require the reference vertex w to be a cut vertex. In the present setting, taking $w = c$ gives

$$A_G(u,v) = A_G(c,v) - A_G(c,u),$$

and Corollary 3.3 makes the two terms on the right-hand side explicit.

Corollary 5.2. *Let*

$$u \in V(G_i) \setminus \{c\}, \qquad v \in V(G_j) \setminus \{c\}, \qquad i \neq j.$$

Then

$$A_G(u,v) = A_{G_j}(c,v) - A_{G_i}(c,u) + 2(M - m_j)R_{G_j}(c,v) - 2(M - m_i)R_{G_i}(c,u).$$

Proof. By Proposition 2.1,

$$A_G(u,v) = A_G(c,v) - A_G(c,u).$$

Applying Corollary 3.3 separately to u and v gives

$$A_G(c,u) = A_{G_i}(c,u) + 2(M - m_i)R_{G_i}(c,u)$$

and

$$A_G(c,v) = A_{G_j}(c,v) + 2(M - m_j)R_{G_j}(c,v).$$

Substitution gives the result. □

What is special in the present setting is that the cut-vertex decomposition gives an explicit formula for the reference quantities $A_G(c,u)$ in terms of local branch data. Consequently, the cross-branch asymmetry admits the explicit representation above.

We next specialize to the case where one or more branches are themselves highly symmetric.

Theorem 5.3. *Suppose that G_i is highly symmetric. Then for every*

$$u \in V(G_i) \setminus \{c\},$$

we have

$$H_G(u,c) = m_i R_{G_i}(c,u)$$

and

$$H_G(c,u) = (2M - m_i) R_{G_i}(c,u).$$

Consequently,

$$A_G(c,u) = 2(M - m_i) R_{G_i}(c,u),$$

and

$$\frac{H_G(c,u)}{H_G(u,c)} = \frac{2M - m_i}{m_i}.$$

Proof. Since G_i is highly symmetric,

$$H_{G_i}(c,u) = H_{G_i}(u,c).$$

By the commute-time identity in G_i,

$$H_{G_i}(c,u) + H_{G_i}(u,c) = 2m_i R_{G_i}(c,u),$$

and therefore

$$H_{G_i}(c,u) = H_{G_i}(u,c) = m_i R_{G_i}(c,u).$$

By Lemma 3.1,

$$H_G(u,c) = m_i R_{G_i}(c,u).$$

Moreover, Theorem 3.2 gives

$$\begin{aligned} H_G(c,u) &= H_{G_i}(c,u) + 2(M - m_i) R_{G_i}(c,u) \\ &= (2M - m_i) R_{G_i}(c,u). \end{aligned}$$

The remaining formulas follow immediately. In particular, the ratio formula also follows directly from Theorem 3.4, because

$$\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)} = 1.$$

□

Combining Corollary 5.2 and Theorem 5.3 gives an explicit description of the asymmetry between two highly symmetric branches.

Corollary 5.4. *Suppose that both G_i and G_j are highly symmetric, where $i \neq j$. For*

$$u \in V(G_i) \setminus \{c\}, \qquad v \in V(G_j) \setminus \{c\},$$

we have

$$A_G(u,v) = 2(M - m_j) R_{G_j}(c,v) - 2(M - m_i) R_{G_i}(c,u).$$

In particular,

$$H_G(u,v) = H_G(v,u)$$

if and only if

$$(M - m_i) R_{G_i}(c,u) = (M - m_j) R_{G_j}(c,v).$$

Proof. Since G_i and G_j are highly symmetric,

$$A_{G_i}(c,u) = 0$$

and

$$A_{G_j}(c,v) = 0.$$

The result follows directly from Corollary 5.2.

□

Even if all branches are highly symmetric, the resulting graph cannot be highly symmetric by Theorem 4.1. For any two highly symmetric branches, the cross-branch asymmetry is governed by the weighted effective-resistance quantities

$$(M - m_i)R_{G_i}(c,u).$$

We next recover a particularly simple formula when all branches are identical.

Let F be a finite connected highly symmetric graph with at least two vertices, fix a vertex $c \in V(F)$, and set

$$m = |E(F)|.$$

For $s \geq 2$, let $G_s(F,c)$ be the graph obtained from s copies of F by identifying their copies of c into one common vertex, again denoted by c.

Corollary 5.5. *Let $u \neq c$ belong to one of the copies of F in $G_s(F,c)$. Then*

$$H_{G_s(F,c)}(u,c) = mR_F(c,u)$$

and

$$H_{G_s(F,c)}(c,u) = (2s-1)mR_F(c,u).$$

Consequently,

$$A_{G_s(F,c)}(c,u) = 2(s-1)mR_F(c,u),$$

and

$$\frac{H_{G_s(F,c)}(c,u)}{H_{G_s(F,c)}(u,c)} = 2s - 1.$$

Proof. First, $F - c$ is connected. Indeed, if $|V(F)| = 2$, then $F = K_2$ and the assertion is immediate. If $|V(F)| \geq 3$, then Corollary 4.2 shows that F is 2-connected, and hence $F - c$ is connected.

Therefore the s copies of F are precisely the s branches of $G_s(F,c)$ at c. Each branch is isomorphic to F and hence is highly symmetric. Moreover,

$$m_i = m$$

for every branch and

$$M = sm.$$

The conclusion follows immediately from Theorem 5.3.

□

The ratio $2s - 1$ is independent of both the vertex u and the internal structure of the highly symmetric graph F. It depends only on the number of copies identified at the common vertex. This recovers the corresponding hitting-time relation for conjoined highly symmetric graphs studied by Palacios and Markowsky [7].

Finally, we record the consequence for arbitrary one-vertex identifications.

Corollary 5.6. *Let $F_1, \ldots, F_s$, where $s \geq 2$, be pairwise vertex-disjoint finite connected simple graphs, each having at least two vertices. For each i, choose a vertex $c_i \in V(F_i)$, and let G be obtained by identifying $c_1, \ldots, c_s$ into a single common vertex c. Then G is not highly symmetric.*

Proof. Since each F_i has at least two vertices, removing the common vertex c separates the vertices belonging to different graphs F_i. Therefore c is a cut vertex of G. The conclusion follows from Theorem 4.1.

□

More generally, Theorem 3.2 and Corollary 5.2 give explicit formulas for the hitting times and their asymmetry under any one-vertex identification, without requiring the constituent graphs to be highly symmetric.

6 Conclusion

We studied the quantitative symmetry breaking of hitting times at cut vertices in finite connected simple graphs. Starting from known cut-vertex decomposition formulas, we derived an exact local-to-global hitting-time ratio and universal lower bounds. More precisely, if c is a cut vertex with branches $G_1, \ldots, G_s$, then for every $u \in V(G_i) \setminus \{c\}$ we obtained the exact relation

$$\frac{H_G(c,u)}{H_G(u,c)} = \frac{M - m_i}{m_i} + \frac{M}{m_i}\frac{H_{G_i}(c,u)}{H_{G_i}(u,c)}.$$

In particular,

$$\frac{H_G(c,u)}{H_G(u,c)} > \frac{M - m_i}{m_i}.$$

This implies that at least $s - 1$ branches have a fixed direction of hitting-time asymmetry, and that on a branch having the minimum number of edges,

$$\frac{H_G(c,u)}{H_G(u,c)} > s - 1.$$

We also showed that the constant $s - 1$ is asymptotically best possible.

As a consequence, no finite connected simple graph with a cut vertex is highly symmetric. Hence every finite connected highly symmetric graph with at least three vertices is 2-connected, and K_2 is the only finite connected simple highly symmetric graph with a cut edge. These conclusions answer the two questions posed by Palacios and Markowsky.

Finally, we combined the general reversibility identity for hitting times with the cut-vertex formulas to express the asymmetry between vertices lying in different branches explicitly in terms of local branch quantities.

Thus, the main contribution of this paper is not the cut-vertex decomposition itself, but the exact local-to-global ratio formula, the universal bounds, their asymptotic sharpness, and the resulting structural consequences.

7 Open Problems

The results of this paper show that a finite connected simple graph with symmetric hitting times cannot have a cut vertex. In particular, every such graph with at least three vertices must be 2-connected.

This result provides a necessary structural condition for graphs with symmetric hitting times. A broader question concerning the structure of such graphs was raised by Georgakopoulos [4]; see also [5] for related work.

Open Problem 7.1. *Let G be a finite connected simple graph satisfying*

$$H_G(u,v) = H_G(v,u)$$

for all $u, v \in V(G)$.

1. *Must G be regular?*
2. *If G is regular, must it be walk-regular?*

Our results provide an additional necessary structural condition for such graphs, namely the absence of cut vertices. However, this condition alone does not settle the regularity or walk-regularity questions.

Declarations

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