

RESEARCH ARTICLE

Analytic Reformulations, Mellin Moments, Bose–Einstein Variance and Gamma–Zipf Mixtures for a Riemann Zeta Inequality

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ABSTRACT

We develop an integrated analytic and probabilistic framework for a proved real-variable inequality involving three consecutive shifts of the Riemann zeta function. After a change of parameter, the inequality is first placed in a chain of exact formulations: a weighted consecutive-zeta-ratio inequality, a strict three-moment inequality for the Bose–Einstein Mellin kernel, a variance-to-mean domination law for the normalized kernel, and a unit-step decrease condition for a zeta-ratio function. The Mellin formulation then becomes the gateway to genuinely new probabilistic consequences. We prove a Gamma–Zipf mixture representation, derive a strict inverse-moment inequality for the Zipf law, determine all positive integer moments, obtain an exact Hurwitz-zeta Laplace transform, and quantify the large-parameter variance excess. A Zipf differentiation identity further converts the stronger differential condition into a covariance inequality. The resulting framework separates exact equivalences from stronger sufficient conditions and shows how a zeta inequality generates nontrivial moment, mixture, and probabilistic structure.

Keywords: Riemann zeta function; Mellin moments; Bose–Einstein distribution; Gamma mixture; Zipf distribution; variance inequality; covariance; log-convexity.

Mathematics Subject Classification: 11M06, 26D07, 33B15, 60E05, 60E15 (MSC2020).

1 Introduction

For $s > 1$, the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \quad (1)$$

Its inequalities, convexity properties, and shifted ratios have been studied extensively; see, for example, [1–10]. Positive Dirichlet series also possess natural convexity properties [11].

Nantomah and Ravi [12] proved a three-shift zeta inequality for positive integer arguments and asked for its extension to positive real arguments. An earlier preprint by the present author [13] established the real-variable positivity result using tail bounds and zeta-ratio monotonicity. The present paper does not duplicate that analytic proof. Instead, it takes the established positivity theorem as its starting point and asks what analytic reformulations and probabilistic structures are encoded by the inequality.

The paper has two connected layers. The first identifies an exact equivalence architecture linking the original zeta expression to a weighted consecutive-ratio inequality, a strict Mellin-moment inequality, a normalized Bose–Einstein variance inequality, and a unit-step monotonicity condition. The second layer uses the Mellin and variance formulations as a gateway to new consequences: a Gamma–Zipf mixture representation, a strict inverse-moment inequality for the Zipf law, all positive integer moments, a Hurwitz-zeta Laplace transform, and the asymptotic size of the variance excess. Finally, differentiation of

Zipf expectations leads to a covariance criterion for the stronger condition $H'(\alpha) < 0$. This last condition is sufficient for the unit-step inequality but is not asserted to be equivalent to it.

Put $\alpha = z + 1 > 1$ and define

$$\mathcal{F}(\alpha) = (\alpha + 1)\zeta(\alpha)\zeta(\alpha + 2) - \alpha\zeta(\alpha + 1)^2 - \zeta(\alpha)\zeta(\alpha + 1). \quad (2)$$

The analytic input is

$$\mathcal{F}(\alpha) > 0, \quad \alpha > 1. \quad (3)$$

We refer to [13] for the analytic proof. Our concern here is the structure generated by (3).

2 Shifted zeta ratios and the first equivalence

Define

$$Q(\alpha) = \frac{\zeta(\alpha + 1)}{\zeta(\alpha)}, \quad \alpha > 1. \quad (4)$$

Theorem 2.1 (Shifted-ratio equivalence). *For each $\alpha > 1$,*

$$\mathcal{F}(\alpha) > 0 \iff (\alpha + 1)Q(\alpha + 1) - \alpha Q(\alpha) > 1.$$

Proof. Since $\zeta(\alpha)\zeta(\alpha + 1) > 0$, division of $\mathcal{F}(\alpha)$ by this product gives

$$\frac{\mathcal{F}(\alpha)}{\zeta(\alpha)\zeta(\alpha + 1)} = (\alpha + 1)\frac{\zeta(\alpha + 2)}{\zeta(\alpha + 1)} - \alpha\frac{\zeta(\alpha + 1)}{\zeta(\alpha)} - 1.$$

The assertion follows from (4). □

The ratio formulation is useful because it isolates the interaction between two consecutive zeta ratios. The next step is more structural: the same inequality is encoded by the first three moments of a positive Mellin kernel, which permits normalization to a probability law.

3 Bose–Einstein Mellin moments

The classical Mellin representation is

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt, \quad s > 1. \quad (5)$$

For $r \geq 0$ and $\alpha > 1$, define

$$M_r(\alpha) = \int_0^\infty \frac{t^{\alpha+r-1}}{e^t - 1} dt = \Gamma(\alpha + r)\zeta(\alpha + r). \quad (6)$$

Thus

$$M_0 = \Gamma(\alpha)\zeta(\alpha), \quad M_1 = \alpha\Gamma(\alpha)\zeta(\alpha + 1), \quad M_2 = \alpha(\alpha + 1)\Gamma(\alpha)\zeta(\alpha + 2).$$

Theorem 3.1 (Strict moment domination). *For every $\alpha > 1$,*

$$M_0(\alpha)M_2(\alpha) - M_1(\alpha)^2 > M_0(\alpha)M_1(\alpha). \quad (7)$$

Proof. From (6) and the Gamma recurrence $\Gamma(x + 1) = x\Gamma(x)$,

$$\begin{aligned} M_0 &= \Gamma(\alpha)\zeta(\alpha), \\ M_1 &= \Gamma(\alpha + 1)\zeta(\alpha + 1) = \alpha\Gamma(\alpha)\zeta(\alpha + 1), \\ M_2 &= \Gamma(\alpha + 2)\zeta(\alpha + 2) = \alpha(\alpha + 1)\Gamma(\alpha)\zeta(\alpha + 2). \end{aligned}$$

Consequently,

$$\begin{aligned} M_0 M_2 &= \alpha(\alpha+1)\Gamma(\alpha)^2 \zeta(\alpha) \zeta(\alpha+2), \\ M_1^2 &= \alpha^2 \Gamma(\alpha)^2 \zeta(\alpha+1)^2, \\ M_0 M_1 &= \alpha \Gamma(\alpha)^2 \zeta(\alpha) \zeta(\alpha+1). \end{aligned}$$

Subtracting the last two quantities from the first and factoring the common positive factor $\alpha \Gamma(\alpha)^2$ yields

$$\begin{aligned} M_0 M_2 - M_1^2 - M_0 M_1 \\ = \alpha \Gamma(\alpha)^2 \left[(\alpha+1) \zeta(\alpha) \zeta(\alpha+2) - \alpha \zeta(\alpha+1)^2 - \zeta(\alpha) \zeta(\alpha+1) \right]. \end{aligned} \quad (8)$$

Since $\alpha > 1$, we have $\alpha \Gamma(\alpha)^2 > 0$, while the bracket is strictly positive by (3). Hence the left-hand side of (8) is positive, proving $M_0 M_2 - M_1^2 > M_0 M_1$. $\square$

Corollary 3.2 (Quantitative strengthening of log-convexity). *For $\alpha > 1$,*

$$M_0 M_2 - M_1^2 > M_0 M_1 > 0.$$

In particular $M_1^2 < M_0 M_2$.

Proof. The first strict inequality is Theorem 3.1. Each M_r is strictly positive because its defining integrand in (6) is positive for $t > 0$; hence $M_0 M_1 > 0$. Therefore $M_0 M_2 - M_1^2 > 0$, and rearranging gives $M_1^2 < M_0 M_2$. $\square$

Remark 3.3. The usual Cauchy–Schwarz/log-convexity argument for the positive Mellin kernel gives only $M_1^2 \leq M_0 M_2$. Theorem 3.1 identifies a strictly positive excess and quantifies it from below by $M_0 M_1$. Thus the present conclusion is not obtained merely by restating ordinary moment log-convexity.

4 A Bose–Einstein probability law and variance domination

Normalize the kernel in (6) by defining

$$p_\alpha(t) = \frac{1}{\Gamma(\alpha) \zeta(\alpha)} \frac{t^{\alpha-1}}{e^t - 1}, \quad t > 0. \quad (9)$$

Equation (5) shows that p_α integrates to one. Let X_α have density (9). Then

$$\mathbb{E} X_\alpha = \frac{M_1}{M_0} = \alpha \frac{\zeta(\alpha+1)}{\zeta(\alpha)}, \quad \mathbb{E} X_\alpha^2 = \frac{M_2}{M_0} = \alpha(\alpha+1) \frac{\zeta(\alpha+2)}{\zeta(\alpha)}. \quad (10)$$

Theorem 4.1 (Bose–Einstein variance domination). *For every $\alpha > 1$,*

$$\text{Var}(X_\alpha) > \mathbb{E} X_\alpha. \quad (11)$$

More precisely,

$$\text{Var}(X_\alpha) - \mathbb{E} X_\alpha = \frac{M_0 M_2 - M_1^2 - M_0 M_1}{M_0^2} > 0. \quad (12)$$

Proof. First note that all moments used below are finite. Near $t = 0$, $e^t - 1 \sim t$, so the integrand defining M_r behaves like $t^{\alpha+r-2}$, which is integrable because $\alpha > 1$ and $r \geq 0$. As $t \rightarrow \infty$, the factor $(e^t - 1)^{-1}$ decays exponentially. Thus (10) is legitimate.

By the elementary variance identity,

$$\begin{aligned} \text{Var}(X_\alpha) &= \mathbb{E}(X_\alpha^2) - (\mathbb{E} X_\alpha)^2 \\ &= \frac{M_2}{M_0} - \left(\frac{M_1}{M_0} \right)^2 \\ &= \frac{M_0 M_2 - M_1^2}{M_0^2}. \end{aligned}$$

Since $\mathbb{E}X_\alpha = M_1/M_0$, subtraction gives the exact identity

$$\text{Var}(X_\alpha) - \mathbb{E}X_\alpha = \frac{M_0M_2 - M_1^2 - M_0M_1}{M_0^2}.$$

The denominator is strictly positive and, by Theorem 3.1, the numerator is strictly positive. Hence $\text{Var}(X_\alpha) > \mathbb{E}X_\alpha$ for every $\alpha > 1$. $\square$

Corollary 4.2. *The squared coefficient of variation satisfies*

$$\frac{\text{Var}(X_\alpha)}{(\mathbb{E}X_\alpha)^2} > \frac{1}{\mathbb{E}X_\alpha}.$$

Proof. By (10), $\mathbb{E}X_\alpha > 0$. Divide the strict inequality $\text{Var}(X_\alpha) > \mathbb{E}X_\alpha$ from Theorem 4.1 by the positive quantity $(\mathbb{E}X_\alpha)^2$. This gives

$$\frac{\text{Var}(X_\alpha)}{(\mathbb{E}X_\alpha)^2} > \frac{\mathbb{E}X_\alpha}{(\mathbb{E}X_\alpha)^2} = \frac{1}{\mathbb{E}X_\alpha}. \quad \square$$

5 Gamma–Zipf mixture representation

The expansion

$$\frac{1}{e^t - 1} = \sum_{k=1}^{\infty} e^{-kt}, \quad t > 0, \quad (13)$$

reveals a mixture structure. Let

$$f_{\alpha,k}(t) = \frac{k^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-kt}, \quad t > 0,$$

be the density of a Gamma random variable with shape α and rate k .

Proposition 5.1 (Gamma–Zipf representation). *Let K_α have the Zipf law*

$$\Pr(K_\alpha = k) = \frac{k^{-\alpha}}{\zeta(\alpha)}, \quad k = 1, 2, \dots \quad (14)$$

Conditionally on $K_\alpha = k$, let

$$X_\alpha \mid K_\alpha = k \sim \text{Gamma}(\alpha, \text{rate } k). \quad (15)$$

Then the unconditional density of X_α is exactly (9).

Proof. For $t > 0$ the geometric-series identity

$$\frac{1}{e^t - 1} = \frac{e^{-t}}{1 - e^{-t}} = \sum_{k=1}^{\infty} e^{-kt}$$

holds with nonnegative terms. Substituting this into (9) gives

$$\begin{aligned} p_\alpha(t) &= \frac{1}{\Gamma(\alpha)\zeta(\alpha)} \sum_{k=1}^{\infty} t^{\alpha-1} e^{-kt} \\ &= \sum_{k=1}^{\infty} \frac{k^{-\alpha}}{\zeta(\alpha)} \frac{k^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-kt} \\ &= \sum_{k=1}^{\infty} \Pr(K_\alpha = k) f_{\alpha,k}(t). \end{aligned}$$

The weights are nonnegative and sum to one because

$$\sum_{k=1}^{\infty} \frac{k^{-\alpha}}{\zeta(\alpha)} = 1.$$

For each k , $f_{\alpha,k}$ integrates to one: with $u = kt$,

$$\int_0^\infty f_{\alpha,k}(t) dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du = 1.$$

Thus the displayed series is precisely the density of the mixture obtained by first sampling K_α from (14) and then sampling $X_\alpha | K_\alpha = k$ from the Gamma law in (15). Its unconditional density is therefore p_α . $\square$

The conditional moments are

$$\mathbb{E}(X_\alpha | K_\alpha) = \frac{\alpha}{K_\alpha}, \quad \text{Var}(X_\alpha | K_\alpha) = \frac{\alpha}{K_\alpha^2}. \quad (16)$$

The Zipf inverse moments satisfy, for $r \geq 0$,

$$\mathbb{E}_\alpha(K_\alpha^{-r}) = \frac{\zeta(\alpha+r)}{\zeta(\alpha)}. \quad (17)$$

6 A strict inverse-moment inequality for the Zipf law

The mixture representation makes it possible to translate Theorem 4.1 into a purely discrete inequality.

Theorem 6.1 (Zipf inverse-moment inequality). *If K_α has the Zipf distribution (14), then, for every $\alpha > 1$,*

$$\mathbb{E}_\alpha(K_\alpha^{-2}) + \alpha \text{Var}_\alpha(K_\alpha^{-1}) > \mathbb{E}_\alpha(K_\alpha^{-1}). \quad (18)$$

Equivalently,

$$(\alpha+1) \frac{\zeta(\alpha+2)}{\zeta(\alpha)} - \alpha \left(\frac{\zeta(\alpha+1)}{\zeta(\alpha)} \right)^2 > \frac{\zeta(\alpha+1)}{\zeta(\alpha)}. \quad (19)$$

Proof. For a Gamma random variable with shape α and rate k ,

$$\mathbb{E}(X_\alpha | K_\alpha = k) = \frac{\alpha}{k}, \quad \text{Var}(X_\alpha | K_\alpha = k) = \frac{\alpha}{k^2}.$$

Since $0 < K_\alpha^{-1} \leq 1$, all inverse moments needed below are finite. The law of total expectation gives

$$\mathbb{E}X_\alpha = \mathbb{E}[\mathbb{E}(X_\alpha | K_\alpha)] = \alpha \mathbb{E}_\alpha(K_\alpha^{-1}).$$

The law of total variance gives

$$\begin{aligned} \text{Var}(X_\alpha) &= \mathbb{E}[\text{Var}(X_\alpha | K_\alpha)] + \text{Var}(\mathbb{E}[X_\alpha | K_\alpha]) \\ &= \mathbb{E}\left(\frac{\alpha}{K_\alpha^2}\right) + \text{Var}\left(\frac{\alpha}{K_\alpha}\right) \\ &= \alpha \mathbb{E}_\alpha(K_\alpha^{-2}) + \alpha^2 \text{Var}_\alpha(K_\alpha^{-1}). \end{aligned}$$

By Theorem 4.1, $\text{Var}(X_\alpha) > \mathbb{E}X_\alpha$. Substitution of the preceding two identities gives

$$\alpha \mathbb{E}_\alpha(K_\alpha^{-2}) + \alpha^2 \text{Var}_\alpha(K_\alpha^{-1}) > \alpha \mathbb{E}_\alpha(K_\alpha^{-1}).$$

Division by $\alpha > 0$ proves (18).

It remains to derive the zeta form. Directly from (14), for $r \geq 0$,

$$\mathbb{E}_\alpha(K_\alpha^{-r}) = \frac{1}{\zeta(\alpha)} \sum_{k=1}^\infty k^{-(\alpha+r)} = \frac{\zeta(\alpha+r)}{\zeta(\alpha)}.$$

Therefore

$$\text{Var}_\alpha(K_\alpha^{-1}) = \frac{\zeta(\alpha+2)}{\zeta(\alpha)} - \left(\frac{\zeta(\alpha+1)}{\zeta(\alpha)} \right)^2.$$

Substituting these expressions into (18) and collecting the two $\zeta(\alpha+2)/\zeta(\alpha)$ terms gives (19). $\square$

Remark 6.2. The first term in (18) is the mean conditional Gamma variance divided by α , while the second is the contribution of the randomness of the Zipf mixing rate. Thus the strict inequality records a genuine interaction between within-component Gamma variation and between-component Zipf variation.

7 Unit-step monotonicity and a differential covariance criterion

The ratio formulation also has a useful discrete-monotonicity interpretation. Define

$$H(\alpha) = \alpha \left(1 - \frac{\zeta(\alpha+1)}{\zeta(\alpha)} \right), \quad \alpha > 1. \quad (20)$$

Theorem 7.1 (Unit-step equivalence). *For each $\alpha > 1$,*

$$\mathcal{F}(\alpha) > 0 \iff H(\alpha) > H(\alpha+1).$$

Proof. Since $Q(\alpha) = 1 - H(\alpha)/\alpha$,

$$(\alpha+1)Q(\alpha+1) - \alpha Q(\alpha) = 1 + H(\alpha) - H(\alpha+1).$$

The result follows from Theorem 2.1. □

Theorem 7.1 is an exact unit-step statement. It should not be confused with pointwise differential monotonicity. The Gamma–Zipf structure developed above makes the distinction especially transparent. For a suitable function g of K_α , we have the following identity.

Lemma 7.2 (Zipf differentiation identity). *Assume the defining series and its derivative converge absolutely and locally uniformly. Then*

$$\frac{d}{d\alpha} \mathbb{E}_\alpha[g(K_\alpha)] = -\text{Cov}_\alpha(g(K_\alpha), \log K_\alpha).$$

Proof. Write $A(\alpha) = \sum_{k \geq 1} g(k)k^{-\alpha}$ and $Z(\alpha) = \zeta(\alpha)$. Then

$$A'(\alpha) = -\sum_{k \geq 1} g(k)(\log k)k^{-\alpha}, \quad \frac{Z'(\alpha)}{Z(\alpha)} = -\mathbb{E}_\alpha(\log K_\alpha).$$

Differentiating A/Z and simplifying yields the covariance identity. □

Since $H(\alpha) = \alpha \mathbb{E}_\alpha(1 - K_\alpha^{-1})$, the preceding lemma gives the following stronger criterion.

Proposition 7.3 (Differential covariance criterion). *For $\alpha > 1$,*

$$H'(\alpha) = \mathbb{E}_\alpha(1 - K_\alpha^{-1}) - \alpha \text{Cov}_\alpha(1 - K_\alpha^{-1}, \log K_\alpha).$$

Consequently,

$$H'(\alpha) < 0 \iff \alpha \text{Cov}_\alpha(1 - K_\alpha^{-1}, \log K_\alpha) > \mathbb{E}_\alpha(1 - K_\alpha^{-1}).$$

If this covariance inequality holds for every $\alpha > 1$, then H is strictly decreasing and hence the unit-step inequality in Theorem 7.1 follows. The converse is not asserted.

8 Synthesis of the equivalent formulations

Theorem 8.1 (Integrated equivalence theorem). *For each $\alpha > 1$, the following statements are equivalent:*

1. $\mathcal{F}(\alpha) > 0$;
2. $(\alpha+1)Q(\alpha+1) - \alpha Q(\alpha) > 1$;
3. $M_0(\alpha)M_2(\alpha) - M_1(\alpha)^2 > M_0(\alpha)M_1(\alpha)$;
4. $\text{Var}(X_\alpha) > \mathbb{E}X_\alpha$;
5. $H(\alpha) > H(\alpha+1)$.

Moreover, all five statements hold for every $\alpha > 1$ by (3). The condition $H'(\alpha) < 0$, equivalently the covariance inequality in Proposition 7.3, is a stronger sufficient condition and is deliberately excluded from the equivalence list.

Proof. The equivalence of (1) and (2) is Theorem 2.1. The factorization in the proof of Theorem 3.1 gives $(1) \iff (3)$. The exact identity (12) gives $(3) \iff (4)$, and Theorem 7.1 gives $(1) \iff (5)$. The final statement follows from Proposition 7.3. $\square$

This theorem closes the equivalence layer of the paper. The Gamma–Zipf representation and Theorem 6.1 go beyond that layer: they do not merely rename the original inequality, but expose how the variance excess decomposes into within-component Gamma variation and between-component Zipf variation. The remaining sections exploit this probabilistic structure further.

9 Beyond equivalence: quantitative moment structure

Classical positivity of the Mellin kernel gives the Hankel/log-convexity inequality

$$M_0 M_2 - M_1^2 \geq 0.$$

The zeta inequality yields the strictly stronger estimate

$$M_0 M_2 - M_1^2 > M_0 M_1,$$

so the normalized statement is not merely nonnegative variance but strict overdispersion, $\text{Var}(X_\alpha) > \mathbb{E}X_\alpha$. The mixture decomposition sharpens the interpretation further: Theorem 6.1 separates the variance into the expected conditional Gamma variance and the variance generated by the random Zipf rate. This is the point at which the paper moves from equivalent formulations to additional probabilistic consequences that are not visible from ordinary log-convexity alone.

10 Further probabilistic consequences

The Gamma–Zipf representation gives more than the first two moments.

Theorem 10.1 (All positive integer moments). *For every integer $m \geq 1$ and every $\alpha > 1$,*

$$\mathbb{E}(X_\alpha^m) = (\alpha)_m \frac{\zeta(\alpha + m)}{\zeta(\alpha)}, \quad (21)$$

where $(\alpha)_m = \alpha(\alpha + 1) \cdots (\alpha + m - 1)$ is the rising factorial.

Proof. Conditionally on $K_\alpha = k$, the variable X_α is Gamma with shape α and rate k . Hence

$$\mathbb{E}(X_\alpha^m \mid K_\alpha = k) = \frac{(\alpha)_m}{k^m}.$$

Taking expectation with respect to the Zipf law and using $\mathbb{E}_\alpha(K_\alpha^{-m}) = \zeta(\alpha + m)/\zeta(\alpha)$ proves (21). $\square$

Theorem 10.2 (Hurwitz-zeta Laplace transform). *For $s \geq 0$ and $\alpha > 1$, the Laplace transform of X_α is*

$$L_\alpha(s) := \mathbb{E}(e^{-sX_\alpha}) = \frac{\zeta(\alpha, 1+s)}{\zeta(\alpha)}, \quad (22)$$

where $\zeta(\alpha, q)$ denotes the Hurwitz zeta function.

Proof. For a Gamma random variable with shape α and rate k ,

$$\mathbb{E}(e^{-sX_\alpha} \mid K_\alpha = k) = \left(\frac{k}{k+s} \right)^\alpha.$$

Averaging with the Zipf weights gives

$$\begin{aligned} L_\alpha(s) &= \frac{1}{\zeta(\alpha)} \sum_{k=1}^{\infty} k^{-\alpha} \left(\frac{k}{k+s} \right)^\alpha \\ &= \frac{1}{\zeta(\alpha)} \sum_{k=1}^{\infty} (k+s)^{-\alpha} = \frac{\zeta(\alpha, 1+s)}{\zeta(\alpha)}. \end{aligned}$$

All terms are nonnegative, so the conditioning/summation step is justified by Tonelli's theorem. $\square$

Corollary 10.3. For every integer $m \geq 1$,

$$(-1)^m L_\alpha^{(m)}(0) = (\alpha)_m \frac{\zeta(\alpha+m)}{\zeta(\alpha)}.$$

Proof. For $q > 0$, termwise differentiation of the Hurwitz zeta series gives $\partial_q^m \zeta(\alpha, q) = (-1)^m (\alpha)_m \zeta(\alpha+m, q)$. Apply this to (22) with $q = 1+s$ and set $s = 0$. $\square$

11 Asymptotic size of the variance excess

Theorem 11.1 (Large-parameter variance excess). As $\alpha \rightarrow \infty$,

$$\text{Var}(X_\alpha) - \mathbb{E}X_\alpha \sim \frac{\alpha(\alpha-1)}{4} 2^{-\alpha}. \quad (23)$$

Consequently, $\text{Var}(X_\alpha)/\mathbb{E}X_\alpha \rightarrow 1^+$.

Proof. From the exact moment formulas,

$$\text{Var}(X_\alpha) - \mathbb{E}X_\alpha = \frac{\alpha}{\zeta(\alpha)^2} \mathcal{F}(\alpha), \quad (24)$$

where

$$\mathcal{F}(\alpha) = (\alpha+1)\zeta(\alpha)\zeta(\alpha+2) - \alpha\zeta(\alpha+1)^2 - \zeta(\alpha)\zeta(\alpha+1).$$

For fixed $j = 0, 1, 2$, write $\zeta(\alpha+j) = 1 + 2^{-\alpha-j} + R_j(\alpha)$, where $R_j(\alpha) = O(3^{-\alpha})$. Substitution and collection of the terms linear in $2^{-\alpha}$ give

$$\mathcal{F}(\alpha) = \frac{\alpha-1}{4} 2^{-\alpha} + O(\alpha 3^{-\alpha} + \alpha 4^{-\alpha}).$$

Since $\zeta(\alpha) = 1 + O(2^{-\alpha})$, equation (24) yields (23). Moreover $\mathbb{E}X_\alpha = \alpha\zeta(\alpha+1)/\zeta(\alpha) \sim \alpha$, so the variance-to-mean ratio tends to 1 from above. $\square$

Remark 11.2. As $\alpha \rightarrow \infty$, the Zipf mixing law concentrates at $K_\alpha = 1$. Hence X_α approaches the moment behavior of a $\text{Gamma}(\alpha, 1)$ variable, whose variance equals its mean. The theorem quantifies the exponentially small positive correction generated by the remaining Zipf mass.

12 Conclusion

We have organized a real-variable Riemann zeta inequality into a single analytic-probabilistic framework. The ratio, Mellin-moment, variance, and unit-step formulations are exactly equivalent, while the differential Zipf covariance inequality is a stronger sufficient condition. More importantly, the Mellin and variance viewpoints generate consequences beyond equivalence: a Gamma–Zipf mixture representation, a strict inverse-moment inequality for the Zipf law, closed formulas for all positive integer moments, an exact Hurwitz-zeta Laplace transform, and an exponentially small but positive large-parameter variance excess. The present paper shows how the established inequality propagates into moment theory and probability.

13 Open Problems

Open Problem 13.1 (Direct probabilistic proof and quantitative strengthening). Give a proof of $\text{Var}(X_\alpha) > \mathbb{E}X_\alpha$ for every $\alpha > 1$ using only the Gamma–Zipf mixture, stochastic ordering, or another probabilistic argument, without invoking the established three-shift zeta inequality. Equivalently, prove directly from the Zipf law that

$$\mathbb{E}_\alpha(K_\alpha^{-2}) + \alpha \text{Var}_\alpha(K_\alpha^{-1}) > \mathbb{E}_\alpha(K_\alpha^{-1}), \quad \alpha > 1.$$

Beyond positivity, determine a probabilistically natural lower bound for the excess that is sharp, or asymptotically sharp, as $\alpha \rightarrow 1^+$ or $\alpha \rightarrow \infty$.

Open Problem 13.2 (Continuous monotonicity). Determine whether

$$H'(\alpha) < 0, \quad \alpha > 1,$$

holds on the whole interval. By Proposition 7.3, this is equivalent to

$$\alpha \text{Cov}_\alpha(1 - K_\alpha^{-1}, \log K_\alpha) > \mathbb{E}_\alpha(1 - K_\alpha^{-1}).$$

A proof would strengthen the established unit-step decrease $H(\alpha) > H(\alpha + 1)$ to continuous strict monotonicity.

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