

RESEARCH ARTICLE

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ABSTRACT

In this paper, we introduce the notion of a P -semiderivation associated with a prime ideal of an arbitrary ring. We establish several basic properties of P -semiderivations and investigate commutativity conditions for the factor ring R/P . Under suitable differential identities involving commutators and Jordan products, we prove that the factor ring R/P is a commutative integral domain or the image of the P -semiderivation is contained in the prime ideal P . Our results generalize several known commutativity theorems concerning derivations and P -derivations to the broader framework of P -semiderivations.

Keywords: Prime ideal; derivation; semiderivation; P -derivation.**Mathematics Subject Classification:** 16Y30, 16W25 (MSC2020).

1 Introduction

Derivations and their generalizations have been extensively investigated in ring theory because of their close relationship with the structural properties of rings. They also arise naturally in several areas of mathematics and mathematical physics. One of the central problems in this area is to determine conditions under which the existence of a derivation forces a ring to be commutative.

The study of commutativity conditions for prime and semiprime rings admitting derivations was initiated by Posner in [1]. Recall that an additive mapping d is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Posner proved that every nonzero centralizing derivation of a prime ring implies the commutativity of the ring. Since then, numerous authors have established commutativity theorems for prime and semiprime rings admitting derivations, generalized derivations, automorphisms, and other related mappings. Consequently, derivations and their generalizations have become an active area of research in ring theory.

In [2], a significant generalization of derivations was introduced by Bergen under the name of semiderivation. An additive mapping $d : R \rightarrow R$ is called a semiderivation if there exists a function $g : R \rightarrow R$ such that (i) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and (ii) $d(g(x)) = g(d(x))$ hold for all $x, y \in R$. In case g is an identity map of R , then all semiderivations associated with g are merely derivations. Another motivating example the mapping $d = g - id_R$, where g is a ring endomorphism of R distinct from the identity mapping id_R . Then d is a semiderivation with associated map g which is not a derivation. In case R is prime and $d \neq 0$, it has been shown by Chang [3] that g must necessarily be a ring endomorphism. After this development, the definition of semiderivation attracted the attention of many mathematicians. Several articles explored the structure of semiderivations and established commutativity conditions in prime rings with semiderivation.

Motivated by previous studies, researchs have investigated ringcommutativity in broader setting. Some authors have been investigating the commutativity of the factor ring R/P without imposing any conditions

on a ring R , where P is a prime ideal. This is done through various appropriate additive mappings such as derivation, left-multiplier or automorphisms. An ideal P in a ring R is said to be prime if $P \neq R$ and for any ideals A, B in R , $AB \subseteq P$ then $A \subseteq P$ or $B \subseteq P$. A ring R is said to be a prime ring if the zero ideal is a prime ideal (that is, if I, J are ideals such that $IJ = 0$, then $I = 0$ or $J = 0$). equivalently, a ring R is said to be prime if $xRy = (0)$ implies that either $x = 0$ or $y = 0$ where $x, y \in R$. Also, we know that R/P is a prime ring where P is prime ideal of R . The relationships among prime ideals and prime rings are analogous to the relationships between mappings and commutativity. Rather than assuming that R is prime, we investigate the structure of the factor ring R/P , where P is prime ideal of R .

The mapping $d : R \rightarrow R$ is called P -additive if it satisfies $d(x+y) - d(x) - d(y) \in P$ for all $x, y \in R$, where P is a prime ideal of R . P -additive mapping d is called a P -derivation if it satisfies the relation $d(xy) - d(x)y - xd(y) \in P$ for all $x, y \in R$, where P is a prime ideal. For examples and counterexamples regarding the existence of these concepts, refer to [4]. After this definition, numerous authors have contributed to the establishment of the most fundamental theorems in ring theory by incorporating automorphisms or derivatives and proving commutation theorems for P -derivations. Motivated by these developments, we introduce the notion of P -semiderivations associated with prime ideals and investigate several commutativity conditions for the factor ring R/P . Our results extend various classical theorems concerning derivations and P -derivations. In the present paper, we investigate algebraic identities involving P -semiderivations associated with prime ideals of an arbitrary ring R . In [5], Daif and Bell proved that R is semiprime ring, I is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in I$, then R contains a nonzero central ideal

2 Results

Throughout the paper, we will make some extensive use of the basic commutator identities:

$$\begin{aligned} [x, yz] &= y[x, z] + [x, y]z \\ [xy, z] &= [x, z]y + x[y, z] \\ x \circ (yz) &= (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z \\ (xy) \circ z &= x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]. \end{aligned}$$

Definition 2.1. Let R be a ring and P an ideal of R . A mapping $d : R \rightarrow R$ is called a P -semiderivation if there exists a function $g : R \rightarrow R$ such that

- (i) $d(x+y) - d(x) - d(y) \in P$
- (ii) $d(xy) - d(x)g(y) - xd(y) \in P$
- (iii) $d(xy) - d(x)y - g(x)d(y) \in P$
- (iv) $d(g(x)) - g(d(x)) \in P$ hold for all $x, y \in R$.

Example. Let S be a prime ring. Suppose the ring $R = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in S \right\}$. Define maps $d, g : R \rightarrow R$ as follow:

$$d\left(\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \text{ and } g\left(\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\right) = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}.$$

Let

$$P = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in S \right\}.$$

It is obvious that P is a prime ideal. Indeed, consider the map $\varphi : R \rightarrow S$ defined by $\varphi\left(\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\right) = x$.

It is easy to verify that φ is a nonzero surjective map. Moreover $\ker(\varphi) = P$. Therefore, by the First Isomorphism Theorem, $R/P \cong S$. Since S is a prime ring, it follows that R/P is a prime ring. Consequently, P is a prime ideal of R . Then it is easy to verify that d is P -semiderivation associated with a nonzero P -additive g of R . Moreover, $d(P) \subseteq P$ and $g(P) \subseteq P$.

Lemma 2.2. Let R be a ring and P an ideal of R . If $d : R \rightarrow R$ is P -semiderivation associated with a nonzero epimorphism and P -additive g of R such that $d(P) \subseteq P$ and $g(P) \subseteq P$, then the mapping $\tilde{d} : R/P \rightarrow R/P$ defined by $\tilde{d}(\bar{x}) = \overline{d(x)}$ is a P -semiderivation associated with a nonzero epimorphism $\tilde{g}$ of R/P .

Proof. Let us prove that $\tilde{d}$ is well defined. Indeed, For $\bar{x}, \bar{y} \in R/P = \{x + P \mid x \in R\}$ such that $\bar{x} = \bar{y}$, we get $x + P = y + P$, so that $x - y \in P$. Then, using P -additivity and $d(P) \subseteq P$, one obtains $d(x) - d(y) \in P$. We obtain that $\tilde{d}(\bar{y}) = \overline{d(y)} = \overline{d(x)} = \tilde{d}(\bar{x})$.

Moreover, for $\bar{x}, \bar{y} \in R/P$ such that $\bar{x} = \bar{y}$, we get $x + P = y + P$, so that $x - y \in P$. Then, using P -additivity and $g(P) \subseteq P$, one obtains $g(x) - g(y) \in P$. We obtain that $\tilde{g}(\bar{y}) = \overline{g(y)} = \overline{g(x)} = \tilde{g}(\bar{x})$. We get $\tilde{g}$ is well defined.

We now verify other semiderivation properties. Let $\bar{x}, \bar{y} \in R/P$. We obtain that

$$\begin{aligned}\tilde{d}(\bar{x} + \bar{y}) &= \tilde{d}(\overline{x + y}) = \overline{d(x + y)} = \overline{d(x) + d(y)} \\ &= \overline{d(x)} + \overline{d(y)} = \tilde{d}(\bar{x}) + \tilde{d}(\bar{y})\end{aligned}$$

and

$$\begin{aligned}\tilde{d}(\bar{x}\bar{y}) &= \tilde{d}(\overline{xy}) = \overline{d(xy)} = \overline{d(x)y + g(x)d(y)} \\ &= \overline{d(x)y} + \overline{g(x)d(y)} = \tilde{d}(\bar{x})\bar{y} + \tilde{g}(\bar{x})\tilde{d}(\bar{y}).\end{aligned}$$

Similarly, $\tilde{d}(\bar{x}\bar{y}) = \tilde{d}(\bar{x})\tilde{g}(\bar{y}) + \bar{x}\tilde{d}(\bar{y})$ is obtained. Moreover, we have

$$\tilde{d}(\tilde{g}(\bar{x})) = \overline{d(g(x))} = \overline{g(d(x))} = \tilde{g}(\tilde{d}(\bar{x})).$$

We conclude that $\tilde{d}$ is a P -semiderivation associated with a nonzero surjective map $\tilde{g}$ of R/P . This completes the proof. $\square$

Lemma 2.3. Let R be a ring, P a prime ideal of R and d a nonzero P -semiderivation associated with a nonzero epimorphism and P -additive g of R such that $d(P) \subseteq P$, $g(P) \subseteq P$ and $a \in R$. If $ad(x) \in P$, for all $x \in R$, then $a \in P$ or $d(R) \subseteq P$.

Proof. By the hypothesis, we have

$$ad(x) \in P$$

and so

$$\bar{a}\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x \in R.$$

Replacing $\bar{x}$ by $\bar{x}\bar{y}$ in the last equation, we get

$$\bar{a}\tilde{d}(\bar{x}\bar{y}) = \bar{0}, \text{ for all } x \in R.$$

and so

$$\bar{a}\tilde{d}(\bar{x})\bar{y} + \bar{a}\tilde{g}(\bar{x})\tilde{d}(\bar{y}) = \bar{0}.$$

Using the hypothesis, we get

$$\bar{a}\tilde{g}(\bar{x})\tilde{d}(\bar{y}) = \bar{0}, \text{ for all } x, y \in R.$$

Since $\tilde{g}$ is surjective, we have

$$\bar{a}R/P\tilde{d}(\bar{y}) = \bar{0}, \text{ for all } y \in R.$$

Since R/P is prime ring, we find that

$$\bar{a} = 0 \text{ or } \tilde{d}(\bar{y}) = \bar{0}, \text{ for all } y \in R.$$

Hence we conclude that

$$a \in P \text{ or } d(R) \subseteq P.$$

$\square$

Lemma 2.4. Let R be a ring, P a prime ideal of R and d a nonzero P -semiderivation associated with a nonzero epimorphism and P -additive g of R such that $d(P) \subseteq P, g(P) \subseteq P$, then

$$g(xy) - g(x)g(y) \in P, \text{ for all } x, y \in R \text{ or } d(R) \subseteq P.$$

Proof. For all $x, y, z \in R$, by the definition of d , we have

$$d(x(yz)) - d(x)yz - g(x)d(yz) \in P$$

and so

$$d(x(yz)) - d(x)yz - g(x)d(y)z - g(x)g(y)d(z) \in P. \quad (1)$$

On the other hand, we get

$$d((xy)z) - d(xy)z - g(xy)d(z) \in P$$

and so

$$d((xy)z) - d(x)yz - g(x)d(y)z - g(xy)d(z) \in P. \quad (2)$$

Subtracting (1) and (2), we obtain that

$$g(xy)d(z) - g(x)g(y)d(z) \in P$$

and so

$$(g(xy)) - g(x)g(y))d(z) \in P, \text{ for all } x, y, z \in R.$$

Applying Lemma 2.3, we obtain that $g(xy) - g(x)g(y) \in P$, for all $x, y \in R$ or $d(R) \subseteq P$. The proof is completed. $\square$

Corollary 2.5. Let R be a ring, P a prime ideal of R and d a nonzero P -semiderivation associated with a nonzero epimorphism and P -additive g of R such that $d(P) \subseteq P, g(P) \subseteq P$, then

$$\tilde{g}(\overline{xy}) = \tilde{g}(\overline{x})\tilde{g}(\overline{y}), \text{ for all } x, y \in R \text{ or } d(R) \subseteq P.$$

Theorem 2.6. Let R be a ring, P a prime ideal of R . Suppose that R admits a nonzero P -semiderivation d associated with a nonzero epimorphism and P -additive g of R such that $d(P) \subseteq P, g(P) \subseteq P$. If for all $x, y \in R$ the following conditions hold:

i) $d(x \circ y) \in P$,

ii) $d(x \circ y) \pm (x \circ y) \in P$,

iii) $d(x \circ y) \pm [x, y] \in P$,

iv) $d([x, y]) \in P$,

v) $d([x, y]) \pm ([x, y]) \in P$,

vi) $d([x, y]) \pm (x \circ y) \in P$,

then R/P is a commutative integral domain or $d(R) \subseteq P$.

Proof. i) By the hypothesis, we get

$$d(x \circ y) \in P, \text{ for all } x, y \in R,$$

and so

$$\tilde{d}(\overline{x \circ y}) = \overline{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\overline{xy}$ in the last equation and using this, we have

$$\tilde{d}(\overline{x})\tilde{g}(\overline{x \circ y}) = \overline{0}, \text{ for all } x, y \in R.$$

Since $\tilde{g}$ is an automorphism, we get

$$\tilde{d}(\overline{x})\tilde{g}(\overline{x})\tilde{g}(\overline{y}) + \tilde{d}(\overline{x})\tilde{g}(\overline{y})\tilde{g}(\overline{x}) = \overline{0}, \text{ for all } x, y \in R.$$

By the surjective $\tilde{g}$, we have

$$\tilde{d}(\bar{x})\tilde{g}(\bar{x})\bar{y} + \tilde{d}(\bar{x})\bar{y}\tilde{g}(\bar{x}) = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{y}\tilde{g}(\bar{z})$ in the last equation and using this equation, we have

$$\tilde{d}(\bar{x})\bar{y}[\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}, \text{ for all } x, y, z \in R. \quad (3)$$

Since R/P is a prime ring, we have

$$\tilde{d}(\bar{x}) = \bar{0} \text{ or } [\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}, \text{ for all } x, z \in R.$$

Let $L = \{\bar{x} \in R/P \mid \tilde{d}(\bar{x}) = \bar{0} \text{ for all } \bar{x} \in R/P\}$ and $K = \{\bar{x} \in R/P \mid [\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}, \text{ for all } z \in R.\}$. Clearly each of L and K is additive subgroup of R/P such that $R/P = L \cup K$. But, a group cannot be the set-theoretic union of two proper subgroups. Hence $L = R/P$ or $K = R/P$. If $K = R/P$, then $[\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}$ for all $x, z \in R$. By the surjective $\tilde{g}$, we have $[\bar{x}, \bar{z}] = \bar{0}$ for all $x, z \in R$. We have R/P is a commutative integral domain. If $L = R/P$, then $\tilde{d}(\bar{x}) = \bar{0}$ for all $\bar{x} \in R/P$. We conclude that $d(R) \subseteq P$. This completes the proof.

ii) Assume that $d(x \circ y) \pm x \circ y \in P$, for all $x, y \in R$. This implies that

$$\tilde{d}(\bar{x} \circ \bar{y}) \pm \bar{x} \circ \bar{y} = \bar{0}, \text{ for all } x, y \in R.$$

Taking $\bar{y}\bar{x}$ instead of $\bar{y}$ in this equation and using this equation, we have

$$\bar{0} = \tilde{d}(\bar{x} \circ \bar{y}\bar{x}) \pm \bar{x} \circ \bar{y}\bar{x} = \tilde{d}(\bar{x} \circ \bar{y})\bar{x} + \tilde{g}(\bar{x} \circ \bar{y})\tilde{d}(\bar{x}) \pm (\bar{x} \circ \bar{y})\bar{x} = \tilde{g}(\bar{x} \circ \bar{y})\tilde{d}(\bar{x}).$$

By the hypothesis, we get

$$\tilde{g}(\bar{x} \circ \bar{y})\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{y}\bar{z}$ in the last equation and using last equation, we have

$$\tilde{g}([\bar{x}, \bar{y}])\tilde{g}(\bar{z})\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R.$$

By the surjective $\tilde{g}$, we have

$$\tilde{g}([\bar{x}, \bar{y}])\bar{z}\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R..$$

Arguing in the similar manner as in the proof of equation (3), we get the result.

iii) By our hypothesis, we have

$$d(x \circ y) \pm [x, y] \in P, \text{ for all } x, y \in R.$$

That is

$$\tilde{d}(\bar{x} \circ \bar{y}) \pm [\bar{x}, \bar{y}] = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{y}\bar{x}$ in the last equation and using last equation, we have

$$\tilde{d}(\bar{x} \circ \bar{y})\bar{x} + \tilde{g}(\bar{x} \circ \bar{y})\tilde{d}(\bar{x}) \pm [\bar{x}, \bar{y}]\bar{x} = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{y}\bar{z}$ in the last equation and using last equation, we have

$$\tilde{g}([\bar{x}, \bar{y}])\tilde{g}(\bar{z})\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R.$$

By the surjective $\tilde{g}$, we have

$$\tilde{g}([\bar{x}, \bar{y}])\bar{z}\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R..$$

Arguing in the similar manner as in the proof of equation (3), we get the result.

iv) By the hypothesis, we have

$$d([x, y]) \in P, \text{ for all } x, y \in R,$$

and so

$$\tilde{d}([\bar{x}, \bar{y}]) = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{x}\bar{y}$ in the last equation and using this, we have

$$\tilde{d}(\bar{x})\tilde{g}([\bar{x}, \bar{y}]) = \bar{0}, \text{ for all } x, y \in R. \quad (4)$$

Replacing $\bar{y}$ by $\bar{y}\bar{z}$ in the last equation and using last equation, we have

$$\tilde{d}(\bar{x})\tilde{g}(\bar{y})\tilde{g}([\bar{x}, \bar{z}]) = \bar{0}, \text{ for all } x, y, z \in R.$$

Since $\tilde{g}$ is an automorphism, we get

$$\tilde{d}(\bar{x})\bar{y}[\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}, \text{ for all } x, y, z \in R. \quad (5)$$

Since R/P is a prime ring, we have

$$\tilde{d}(\bar{x}) = \bar{0} \text{ or } [\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}, \text{ for all } x, z \in R.$$

Let $L = \{\bar{x} \in R/P \mid \tilde{d}(\bar{x}) = \bar{0} \text{ for all } \bar{x} \in R/P\}$ and $K = \{\bar{x} \in R/P \mid [\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}, \text{ for all } z \in R.\}$. Clearly each of L and K is additive subgroup of R/P such that $R/P = L \cup K$. But, a group cannot be the set-theoretic union of two proper subgroups. Hence $L = R/P$ or $K = R/P$. If $K = R/P$, then $[\tilde{g}(\bar{x}), \tilde{g}(\bar{z})] = \bar{0}$ for all $x, z \in R$. By the surjective $\tilde{g}$, we have $[\bar{x}, \bar{z}] = \bar{0}$ for all $x, z \in R$. We have R/P is a commutative integral domain. If $L = R/P$, then $\tilde{d}(\bar{x}) = \bar{0}$ for all $\bar{x} \in R/P$. We conclude that $d(R) \subseteq P$. This completes the proof.

v) Assume that $d([x, y]) \pm [x, y] \in P$, for all $x, y \in R$. This implies that

$$\tilde{d}([\bar{x}, \bar{y}]) \pm [\bar{x}, \bar{y}] = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}\bar{x}$ instead of $\bar{y}$ in this equation and using this equation, we have

$$\bar{0} = \tilde{d}([\bar{x}, \bar{y}\bar{x}]) \pm [\bar{x}, \bar{y}\bar{x}] = \tilde{d}([\bar{x}, \bar{y}])\bar{x} + \tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}) \pm ([\bar{x}, \bar{y}])\bar{x} = \tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}).$$

By the hypothesis, we get

$$\tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{y}\bar{z}$ in the last equation and using last equation, we have

$$\tilde{g}([\bar{x}, \bar{y}])\tilde{g}(\bar{z})\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R.$$

By the surjective $\tilde{g}$, we have

$$\tilde{g}([\bar{x}, \bar{y}])\bar{z}\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R.$$

Arguing in the similar manner as in the proof of equation (3), we get the result.

vi) By our hypothesis, we have

$$d([x, y]) \pm (x \circ y) \in P, \text{ for all } x, y \in R.$$

That is

$$\tilde{d}([\bar{x}, \bar{y}]) \pm \bar{x} \circ \bar{y} = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{y}\bar{x}$ in the last equation and using last equation, we have

$$\tilde{d}([\bar{x}, \bar{y}])\bar{x} + \tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}) \pm (\bar{x} \circ \bar{y})\bar{x} = \bar{0}, \text{ for all } x, y \in R.$$

Replacing $\bar{y}$ by $\bar{y}\bar{z}$ in the last equation and using last equation, we have

$$\tilde{g}([\bar{x}, \bar{y}])\tilde{g}(\bar{z})\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R.$$

By the surjective $\tilde{g}$, we have

$$\tilde{g}([\bar{x}, \bar{y}])\bar{z}\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y, z \in R.$$

Arguing in the similar manner as in the proof of equation (3), we get the result. $\square$

Theorem 2.7. Let R be a ring, P a prime ideal of R . Suppose that R admits a nonzero P -semiderivation d associated with a nonzero epimorphism and P -additive g of R such that $d(P) \subseteq P, g(P) \subseteq P$. If for all $x, y \in R$ the following conditions hold:

$$i) d([x, y]) \pm [d(x), y] \in P,$$

$$ii) d([x, y]) \pm [x, d(y)] \in P,$$

$$iii) d(x \circ y) \pm d(x) \circ y \in P,$$

then R/P is a commutative integral domain or $d(R) \subseteq P$.

Proof. i) By the hypothesis

$$d([x, y]) \pm [d(x), y] \in P, \text{ for all } x, y \in R.$$

That is

$$\tilde{d}([\bar{x}, \bar{y}]) \pm [\tilde{d}(\bar{x}), \bar{y}] = \bar{0}, \text{ for all } x, y \in R.$$

Taking yx instead of y in the hypothesis and using this, we have

$$\begin{aligned} \bar{0} &= \tilde{d}([\bar{x}, \bar{y}]\bar{x}) \pm [\tilde{d}(\bar{x}), \bar{y}\bar{x}], \\ &= \tilde{d}([\bar{x}, \bar{y}])\bar{x} + \tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}) \pm [\tilde{d}(\bar{x}), \bar{y}]\bar{x} \pm \bar{y}[\tilde{d}(\bar{x}), \bar{x}] \\ &= \tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}) \pm \bar{y}[\tilde{d}(\bar{x}), \bar{x}]. \end{aligned}$$

Noting that $[\tilde{d}(\bar{x}), \bar{x}] = \bar{0}$, for all $x \in R$ by the hypothesis. Then the last equation gives that

$$\tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y \in R.$$

Arguing in the similar manner as in the equation (3), we get the result.

ii) By the hypothesis, we have

$$d([x, y]) \pm [x, d(y)] \in P, \text{ for all } x, y \in R.$$

That is

$$\tilde{d}([\bar{x}, \bar{y}]) \pm [\bar{x}, \tilde{d}(\bar{y})] = \bar{0}, \text{ for all } x, y \in R.$$

Taking yx instead of x in the hypothesis and using this, we have

$$\begin{aligned} \bar{0} &= \tilde{d}([\bar{y}\bar{x}, \bar{y}]) \pm [\bar{y}\bar{x}, \tilde{d}(\bar{y})] \\ &= \tilde{d}(\bar{y}[\bar{x}, \bar{y}]) \pm \bar{y}[\bar{x}, \tilde{d}(\bar{y})] \pm [\bar{y}, \tilde{d}(\bar{y})]\bar{x} \\ &= \tilde{d}(\bar{y})\tilde{g}([\bar{x}, \bar{y}]) + \bar{y}\tilde{d}([\bar{x}, \bar{y}]) \pm \bar{y}[\bar{x}, \tilde{d}(\bar{y})] \pm [\bar{y}, \tilde{d}(\bar{y})]\bar{x} \\ &= \tilde{d}(\bar{y})\tilde{g}([\bar{x}, \bar{y}]) \pm [\bar{y}, \tilde{d}(\bar{y})]\bar{x} \end{aligned}$$

Noting that $[\tilde{d}(\bar{y}), \bar{y}] = \bar{0}$, for all $y \in R$ by the hypothesis. We have

$$\tilde{d}(\bar{y})\tilde{g}([\bar{x}, \bar{y}]) = \bar{0}, \text{ for all } x, y \in R.$$

Arguing in the similar manner as in the equation (3).

iii) By the hypothesis, we get

$$d(x \circ y) \pm d(x) \circ y \in P, \text{ for all } x, y \in R.$$

That is

$$\tilde{d}(\bar{x} \circ \bar{y}) \pm \tilde{d}(\bar{x}) \circ \bar{y} = \bar{0}, \text{ for all } x, y \in R.$$

Taking yx instead of y in the hypothesis and using this, we have

$$\begin{aligned} \bar{0} &= \tilde{d}((\bar{x} \circ \bar{y})\bar{x}) \pm \tilde{d}(\bar{x}) \circ \bar{y}\bar{x} \\ &= \tilde{d}(\bar{x} \circ \bar{y})\bar{x} + \tilde{g}(\bar{x} \circ \bar{y})\tilde{d}(\bar{x}) \pm (\tilde{d}(\bar{x}) \circ \bar{y})\bar{x} \pm \bar{y}[\tilde{d}(\bar{x}), \bar{x}] \\ &= \tilde{g}(\bar{x} \circ \bar{y})\tilde{d}(\bar{x}) \pm \bar{y}[\tilde{d}(\bar{x}), \bar{x}]. \end{aligned}$$

Replacing y by zy in the last equation and using this equation, we have

$$\begin{aligned} \bar{0} &= \tilde{g}(\bar{z})\tilde{g}(\bar{x} \circ \bar{y})\tilde{d}(\bar{x}) + \tilde{g}([\bar{x}, \bar{z}])\tilde{g}(\bar{y})\tilde{d}(\bar{x}) \pm \bar{z}\bar{y}[\tilde{d}(\bar{x}), \bar{x}] \\ &= \tilde{g}([\bar{x}, \bar{z}])\tilde{g}(\bar{y})\tilde{d}(\bar{x}) \end{aligned}$$

$$\tilde{g}([\bar{x}, \bar{y}])\tilde{d}(\bar{x}) = \bar{0}, \text{ for all } x, y \in R.$$

Arguing in the similar manner as in the equation (3), we get the result. $\square$

3 Open Problem

It would be of interest to investigate whether the commutativity results obtained in this paper remain valid when prime ideals are replaced by semiprime ideals or more general classes of ideals. Furthermore, one may introduce and study generalized P -semiderivations, Jordan P -semiderivations, and multiplicative P -semiderivations, and examine whether the corresponding center-like subsets still characterize the center modulo P . In addition, it would be worthwhile to determine whether these results can be established under weaker torsion assumptions or extended to non-associative algebraic structures such as Lie rings and Jordan rings.

Declarations

Conflicts of interest. The authors have no conflicts of interest to declare that are relevant to the content of this article.

Ethics approval. Not applicable.

Data availability. No datasets were generated or analyzed during this study.

References

- [1] E. C. Posner, "Derivations in prime rings," *Proc. Amer. Math. Soc.*, vol. 8, pp. 1093–1100, 1957, doi: 10.2307/2032686.
- [2] J. Bergen, "Derivations in prime rings," *Canad. Math. Bull.*, vol. 26, pp. 267–270, 1983, doi: 10.4153/CMB-1983-042-2.
- [3] J. C. Chang, "On semiderivations of prime rings," *Chinese J. Math.*, vol. 12, pp. 255–262, 1984.
- [4] G. S. Sandhu, A. Boua, and N. Ur Rehman, "Some results involving P -derivations and prime ideals in rings," *Annali dell'Università di Ferrara*, vol. 69, pp. 587–604, 2023, doi: 10.1007/s11565-022-00453-2.
- [5] M. N. Daif and H. E. Bell, "Remarks on derivations on semiprime rings," *Int. J. Math. Math. Sci.*, vol. 15, no. 1, pp. 205–206, 1992, doi: 10.1155/S016117292000255.