

RESEARCH ARTICLE

Extending the Scope of the Nesbitt Inequality: A Parametric and Functional Approach

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Received: 10 March 2026 Accepted: 3 September 2026 Published: 6 October 2026

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Publisher: Al-Zaytoonah University of Jordan (ZUJ), Amman, Jordan.

Cite this article: C. Chesneau, "Extending the scope of the Nesbitt inequality: A parametric and functional approach," *Int. J. Open Problems Compt. Math.*, vol. 19, no. 3, 2026, doi: <https://doi.org/10.15849/ijopcm.v19i4.99>

ABSTRACT

This paper makes three main contributions to the study of the Nesbitt inequality. First, we revisit a three-variable parametric extension and present a more transparent proof of a known result. Second, building on this proof, we generalize the inequality to the multivariable setting and identify acceptable parameter constraints. Third, we further broaden this framework using convex and increasing functions. Finally, we formulate an open problem regarding the necessary and sufficient conditions on the involved parameters.

Keywords: Nesbitt's inequality; power mean inequality; Radon's inequality; rearrangement inequality.

Mathematics Subject Classification: 26D15 (MSC2020).

1 Introduction

The classical Nesbitt inequality, proposed by A. M. Nesbitt in 1903 [1], is one of the most foundational cyclic rational inequalities in real analysis. It asserts that, for any variables $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

The equality occurs if $a = b = c$. Over the past century, this simple yet rich inequality has served as a benchmark for a wide array of analytical methods, including the Titu lemma, the arithmetic mean-geometric mean inequality, the Radon inequality, and the Jensen inequality.

Due to its versatile structure, numerous researchers have sought to extend, refine, and generalize the Nesbitt inequality along various analytical directions. A natural direction involves increasing the number of variables, yielding cyclic sums over n terms, such as those investigated by H. S. Shapiro concerning the behavior of cyclic rational sums in higher dimensions [2–4]. Another prominent direction focuses on parameterizing the numerators and denominators, or incorporating power weights and deformation parameters, to produce broader continuous families of inequalities [4–10], alongside recently developed integral representation techniques [11]. Such parameterizations reveal acceptable boundary conditions under which functional bounds remain valid. As an example that has motivated this paper, [9, Theorem 2.1] established that, for any variables $a, b, c > 0$ and parameters $\alpha, \beta > 0$,

$$\frac{a}{\alpha b + \beta c} + \frac{b}{\alpha c + \beta a} + \frac{c}{\alpha a + \beta b} \geq \frac{3}{\alpha + \beta}. \quad (1)$$

The equality occurs if $a = b = c$. Setting $\alpha = \beta = 1$ immediately recovers the Nesbitt inequality.

Furthermore, unifying algebraic inequalities within abstract functional frameworks, specifically by employing convex and monotonic functions, enables the derivation of new classes of inequalities with broad applications in approximation theory and functional analysis. See [12] for Nesbitt-type inequalities

with applications to the Riemann zeta function, and see [13] for a general framework involving convex cyclic inequalities.

Motivated by these ongoing developments, the primary objective of this paper is to build a unified, multi-parameter, and functional framework for Nesbitt-type inequalities. First, we discuss a multi-parameter extension of the classic three-variable Nesbitt inequality examined by [10]. It consists in inserting tuning parameters into the denominator structures, providing lower bounds that continuously reduce to the classic $3/2$ bound as a special case. A new comprehensive proof is proposed. Second, building on this three-variable setup, we generalize the inequality to n variables $x_1, x_2, \dots, x_n > 0$ and identify acceptable parameter constraints required for a valid extension. Third, we further broaden this framework by replacing explicit linear terms with a broader class of convex and increasing functions, yielding unified functional extensions from which several known and new cyclic inequalities emerge.

The remainder of this paper is organized into several dedicated sections. Section 2 discusses the parametric three-variable Nesbitt-type inequality. Section 3 extends this result to the multivariable setting. Section 4 develops the general functional framework incorporating convex increasing functions. Section 5 offers some concluding remarks. Finally, an open problem is formulated in Section 6.

2 Parametric three-variable Nesbitt-type inequality

2.1 A key result

In this subsection, we establish a connection between two existing results: Equation (1) (i.e., [9, Theorem 2.1]) and [10, Theorem 7].

To begin, the inequality in Equation (1) inspires the following open question: is it true that

$$\frac{a}{\alpha a + \beta b + \gamma c} + \frac{b}{\alpha b + \beta c + \gamma a} + \frac{c}{\alpha c + \beta a + \gamma b} \geq \frac{3}{\alpha + \beta + \gamma} \quad (2)$$

holds for any $a, b, c > 0$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma > 0$?

While Equation (1) confirms an affirmative answer when $\alpha = 0$, the answer is negative in general for $\alpha \neq 0$. As a counterexample, consider the parameter choice $(\alpha, \beta, \gamma) = (1, 1, 0)$, for which the right-hand side equals $3/2$. Setting $(a, b, c) = (1, x, x^2)$ for $x > 1$, the left-hand side simplifies to

$$\frac{1}{1+x} + \frac{x}{x+x^2} + \frac{x^2}{x^2+1} = \frac{2}{1+x} + \frac{x^2}{x^2+1}.$$

Taking the limit as $x \rightarrow \infty$ yields 1, which is strictly less than $3/2$. Indeed, for a concrete value such as $x = 10$, the left-hand side evaluates to approximately $1.1719 < 1.5$, demonstrating that the inequality fails when $\alpha \neq 0$. This raises the following question: can we find an acceptable parameter domain for which the inequality in Equation (2) holds?

This question is answered in [10, Theorem 7], from which we obtain the following statement.

Theorem 2.1. *Let $\alpha, \beta, \gamma \geq 0$ be parameters such that $\alpha + \beta + \gamma > 0$ and*

$$2\alpha \leq \beta + \gamma.$$

For any $a, b, c > 0$, the following cyclic inequality holds:

$$\frac{a}{\alpha a + \beta b + \gamma c} + \frac{b}{\alpha b + \beta c + \gamma a} + \frac{c}{\alpha c + \beta a + \gamma b} \geq \frac{3}{\alpha + \beta + \gamma}.$$

The equality occurs if $a = b = c$, or $\alpha = \beta = \gamma$.

Thus, this theorem establishes sufficient parameter conditions, namely, $2\alpha \leq \beta + \gamma$, under which the inequality in Equation (2) holds. However, these conditions do not fully characterize the entire admissible parameter domain. This is illustrated by the choice $\alpha = 1$ and $\beta = \gamma = 0$, where the inequality remains valid (holding with equality) despite violating $2\alpha \leq \beta + \gamma$.

Several examples illustrating the utility of Theorem 2.1 are presented in the following section.

2.2 Examples

Isolating the variables in the denominator completely from their respective numerators, we set $\alpha = 0$ and $\beta = \gamma = 1$. The parameter constraint $2(0) \leq 1 + 1$ is satisfied, and Theorem 2.1 directly yields

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{0+1+1} = \frac{3}{2}.$$

We recognize the Nesbitt inequality.

The case $\alpha = \beta = \gamma = 1$ is degenerate, as it reduces to the trivial inequality $a + b + c \geq a + b + c$.

As another example, setting $\alpha = 1$, $\beta = 2$, and $\gamma = 1$, the parameter constraint $2(1) \leq 2 + 1$ is satisfied, and Theorem 2.1 directly yields

$$\frac{a}{a+2b+c} + \frac{b}{b+2c+a} + \frac{c}{c+2a+b} \geq \frac{3}{4}.$$

This can be viewed as an extension of the Nesbitt inequality in which no variable is omitted from the denominators.

2.3 On the original proof

In [10], the proof of Theorem 2.1 relies on the Cauchy-Schwarz inequality (or the Titu lemma) combined with a direct algebraic expansion. For completeness, we summarize below the steps leading to the main inequality.

Applying the Cauchy-Schwarz inequality, we have

$$\left(\frac{a}{\alpha a + \beta b + \gamma c} + \frac{b}{\alpha b + \beta c + \gamma a} + \frac{c}{\alpha c + \beta a + \gamma b} \right) \times (a(\alpha a + \beta b + \gamma c) + b(\alpha b + \beta c + \gamma a) + c(\alpha c + \beta a + \gamma b)) \geq (a + b + c)^2.$$

It follows that

$$\begin{aligned} & \frac{a}{\alpha a + \beta b + \gamma c} + \frac{b}{\alpha b + \beta c + \gamma a} + \frac{c}{\alpha c + \beta a + \gamma b} \\ & \geq \frac{(a + b + c)^2}{a(\alpha a + \beta b + \gamma c) + b(\alpha b + \beta c + \gamma a) + c(\alpha c + \beta a + \gamma b)}. \end{aligned}$$

On the other hand, direct calculations give

$$\begin{aligned} & \frac{(a + b + c)^2}{a(\alpha a + \beta b + \gamma c) + b(\alpha b + \beta c + \gamma a) + c(\alpha c + \beta a + \gamma b)} \\ & = \frac{3}{\alpha + \beta + \gamma} \\ & + \frac{(\beta + \gamma - 2\alpha)(a^2 + b^2 + c^2 - ab - bc - ca)}{(\alpha + \beta + \gamma)[a(\alpha a + \beta b + \gamma c) + b(\alpha b + \beta c + \gamma a) + c(\alpha c + \beta a + \gamma b)]} \\ & = \frac{3}{\alpha + \beta + \gamma} \\ & + \frac{(\beta + \gamma - 2\alpha)[(a - b)^2 + (b - c)^2 + (c - a)^2]}{2(\alpha + \beta + \gamma)[a(\alpha a + \beta b + \gamma c) + b(\alpha b + \beta c + \gamma a) + c(\alpha c + \beta a + \gamma b)]} \\ & \geq \frac{3}{\alpha + \beta + \gamma}. \end{aligned} \tag{3}$$

This concludes the main inequality of the theorem.

While the first equality in Equation (3) is straightforward in principle, verifying it requires some tedious algebraic manipulation. Furthermore, this form does not immediately reveal how the factor $\beta + \gamma - 2\alpha$, which governs the condition $2\alpha \leq \beta + \gamma$, naturally emerges.

On the other hand, a major advantage of this proof is that its explicit structure allows for a precise characterization of the equality cases: under the condition $2\alpha \leq \beta + \gamma$, equality holds if and only if $a = b = c$ or $\alpha = \beta = \gamma$.

2.4 An alternative proof

In this subsection, to provide better insight and facilitate future extensions, we introduce an alternative proof approach. While it still relies on the Titu lemma, it proceeds via a development based on cross-multiplication, coefficient regrouping, and sum-of-squares factorization.

Alternative proof of the main inequality in Theorem 2.1. By multiplying the numerator and denominator of each fractional term by a , b , and c , respectively, and applying the Titu lemma, we obtain

$$\begin{aligned} & \frac{a}{\alpha a + \beta b + \gamma c} + \frac{b}{\alpha b + \beta c + \gamma a} + \frac{c}{\alpha c + \beta a + \gamma b} \\ &= \frac{a^2}{\alpha a^2 + \beta ab + \gamma ac} + \frac{b^2}{\alpha b^2 + \beta bc + \gamma ab} + \frac{c^2}{\alpha c^2 + \beta ac + \gamma bc} \\ &\geq \frac{(a+b+c)^2}{\alpha a^2 + \beta ab + \gamma ac + \alpha b^2 + \beta bc + \gamma ab + \alpha c^2 + \beta ac + \gamma bc} \\ &= \frac{(a+b+c)^2}{\alpha(a^2 + b^2 + c^2) + (\beta + \gamma)(ab + bc + ca)}. \end{aligned}$$

Building on this result, we aim to show that

$$\frac{(a+b+c)^2}{\alpha(a^2 + b^2 + c^2) + (\beta + \gamma)(ab + bc + ca)} \geq \frac{3}{\alpha + \beta + \gamma}.$$

Since $a, b, c > 0$ and the parameters are non-negative, the denominators are strictly positive. Thus, we can cross-multiply to obtain the following equivalent inequality:

$$(\alpha + \beta + \gamma)(a+b+c)^2 \geq 3\alpha(a^2 + b^2 + c^2) + 3(\beta + \gamma)(ab + bc + ca).$$

Using the algebraic identity $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we obtain

$$\begin{aligned} & (\alpha + \beta + \gamma)(a^2 + b^2 + c^2) + 2(\alpha + \beta + \gamma)(ab + bc + ca) \\ & \geq 3\alpha(a^2 + b^2 + c^2) + 3(\beta + \gamma)(ab + bc + ca). \end{aligned}$$

Subtracting the right-hand side from the left-hand side, we group the coefficients for $(a^2 + b^2 + c^2)$ and $(ab + bc + ca)$, as follows:

$$[\alpha + \beta + \gamma - 3\alpha](a^2 + b^2 + c^2) + [2\alpha + 2\beta + 2\gamma - 3\beta - 3\gamma](ab + bc + ca) \geq 0.$$

Simplifying the coefficients yields

$$(\beta + \gamma - 2\alpha)(a^2 + b^2 + c^2) + (2\alpha - \beta - \gamma)(ab + bc + ca) \geq 0.$$

Factoring out $(\beta + \gamma - 2\alpha)$, we arrive at

$$(\beta + \gamma - 2\alpha)(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0.$$

By the initial hypothesis, $2\alpha \leq \beta + \gamma$, we have $(\beta + \gamma - 2\alpha) \geq 0$. Moreover, it is well-known that $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$ (which derives from $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$). Since the product of two non-negative terms is non-negative, the inequality holds true. This concludes the proof of the theorem. $\square$

In key respects, this proof offers a more transparent alternative to the one in [10]. In particular, by combining the Titu lemma with elementary algebraic expansion and coefficient grouping, it naturally reduces the core expression to $(\beta + \gamma - 2\alpha)(a^2 + b^2 + c^2 - ab - bc - ca)$, thereby providing a clear derivation of the parameter condition $2\alpha \leq \beta + \gamma$. Owing to its transparency, this methodology provides a flexible framework for a multivariable generalization, as explored in the next section.

3 Extension to the multivariable setting

3.1 Main result

The multivariable generalization of Theorem 2.1 is presented in the following theorem. Notably, the inclusion of an additional parameter p further enhances the generality of the result.

Theorem 3.1. Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$, $p \geq 0$, and $\omega_1, \dots, \omega_n \geq 0$ be parameters such that $\sum_{k=1}^n \omega_k > 0$ and, for any $j = 2, \dots, n$,

$$\omega_j + \omega_{n-j+2} \geq \frac{2}{n} \sum_{k=1}^n \omega_k. \quad (4)$$

For any $x_1, \dots, x_n > 0$ with the indices taken cyclically modulo n (i.e., $x_{n+k} = x_k$ for any $k = 1, \dots, n$), the following cyclic inequality holds:

$$\sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} \geq \frac{n}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^p.$$

The equality occurs if $x_1 = \dots = x_n$, or $p = 0$ and $\omega_1 = \dots = \omega_n$.

Proof. By multiplying the numerator and denominator by x_i and applying the Titu lemma, we get

$$\sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} = \sum_{i=1}^n \frac{x_i^{p+2}}{x_i \sum_{j=1}^n \omega_j x_{i+j-1}} \geq \frac{\left(\sum_{i=1}^n x_i^{\frac{p+2}{2}} \right)^2}{\sum_{i=1}^n x_i \sum_{j=1}^n \omega_j x_{i+j-1}}. \quad (5)$$

We can express the denominator as follows:

$$\sum_{i=1}^n x_i \sum_{j=1}^n \omega_j x_{i+j-1} = \sum_{i=1}^n \sum_{j=1}^n \omega_j x_i x_{i+j-1} = \sum_{j=1}^n \omega_j C_j, \quad (6)$$

where

$$C_j = \sum_{i=1}^n x_i x_{i+j-1}. \quad (7)$$

Following the alternative proof methodology from Subsection 2.4 for the case $p = 0$, we now aim to establish that

$$\sum_{j=1}^n \omega_j C_j \leq \left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \left(\sum_{i=1}^n x_i \right)^2.$$

To this end, a notable property of C_j is described in the following lemma.

Lemma 3.2. For any $j = 1, \dots, n$, let C_j be defined as in Equation (7). Due to the cyclic symmetry of $x_1, \dots, x_n$, we have

$$C_j = C_{n-j+2}.$$

Proof. By definition, we have $C_k = \sum_{i=1}^n x_i x_{i+k-1}$. Setting $k = n - j + 2$, we get

$$C_{n-j+2} = \sum_{i=1}^n x_i x_{i+(n-j+2)-1} = \sum_{i=1}^n x_i x_{i+n-j+1}.$$

Applying the cyclic symmetry of $x_1, \dots, x_n$, we have $x_{i+n-j+1} = x_{i-j+1}$, so that

$$C_{n-j+2} = \sum_{i=1}^n x_i x_{i-j+1}.$$

Let $m = i - j + 1$, which gives $i = m + j - 1$. As i ranges from 1 to n , m ranges through a complete system of n indices modulo n . Rewriting the term $x_i x_{i-j+1}$ in terms of m , we obtain

$$x_i x_{i-j+1} = x_{m+j-1} x_m.$$

Summing over all n cyclic positions, we get

$$C_{n-j+2} = \sum_{m=1}^n x_{m+j-1} x_m = C_j.$$

This concludes the proof of the lemma. □

The following lemma is a consequence of Lemma 3.2.

Lemma 3.3. For any $j = 1, \dots, n$, let C_j be defined as in Equation (7) and set $\omega_{n+1} = \omega_1$. Due to the cyclic symmetry of $x_1, \dots, x_n$, we have

$$\sum_{j=1}^n \omega_j C_j = \sum_{j=1}^n \left(\frac{\omega_j + \omega_{n-j+2}}{2} \right) C_j.$$

Proof. Applying Lemma 3.2, i.e., $C_j = C_{n-j+2}$, and setting $k = n - j + 2$, we obtain

$$\begin{aligned} \sum_{j=1}^n \left(\frac{\omega_j + \omega_{n-j+2}}{2} \right) C_j &= \frac{1}{2} \sum_{j=1}^n \omega_j C_j + \frac{1}{2} \sum_{j=1}^n \omega_{n-j+2} C_j \\ &= \frac{1}{2} \sum_{j=1}^n \omega_j C_j + \frac{1}{2} \sum_{j=1}^n \omega_{n-j+2} C_{n-j+2} \\ &= \frac{1}{2} \sum_{j=1}^n \omega_j C_j + \frac{1}{2} \sum_{k=1}^n \omega_k C_k = \sum_{j=1}^n \omega_j C_j. \end{aligned}$$

This concludes the proof of the lemma. □

Equipped with Lemma 3.3, we now return to the proof of Theorem 3.1.

For any $j = 1, 2, \dots, n$, we set

$$\bar{\mu}_j = \frac{\omega_j + \omega_{n-j+2}}{2} - \frac{1}{n} \sum_{k=1}^n \omega_k,$$

with $\omega_{n+1} = \omega_1$ (which appears only for $j = 1$).

It follows from Lemma 3.3 and the definition of $\bar{\mu}_j$ that

$$\begin{aligned} \sum_{j=1}^n \omega_j C_j &= \sum_{j=1}^n \left(\frac{\omega_j + \omega_{n-j+2}}{2} \right) C_j \\ &= \sum_{j=1}^n \left(\frac{1}{n} \sum_{k=1}^n \omega_k + \bar{\mu}_j \right) C_j = \left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \sum_{j=1}^n C_j + \sum_{j=1}^n \bar{\mu}_j C_j \\ &= \left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \sum_{j=1}^n C_j + \bar{\mu}_1 C_1 + \sum_{j=2}^n \bar{\mu}_j C_j. \end{aligned} \tag{8}$$

Let us now prove that

$$\bar{\mu}_1 C_1 + \sum_{j=2}^n \bar{\mu}_j C_j \leq 0.$$

By a direct application of the rearrangement inequality, for any $j = 1, \dots, n$, we have

$$\sum_{i=1}^n x_i x_{i+j-1} \leq \sum_{i=1}^n x_i^2,$$

so that

$$C_j \leq C_1.$$

Furthermore, our initial parameter condition in Equation (4) guarantees that, for any $j = 2, \dots, n$,

$$\bar{\mu}_j \geq 0.$$

Therefore, we have

$$\sum_{j=2}^n \bar{\mu}_j C_j \leq \sum_{j=2}^n \bar{\mu}_j C_1 = C_1 \sum_{j=2}^n \bar{\mu}_j.$$

Let us now work on this last sum. Setting $m = n - j + 2$, observe that

$$\begin{aligned} \sum_{j=1}^n \bar{\mu}_j &= \sum_{j=1}^n \left(\frac{\omega_j + \omega_{n-j+2}}{2} - \frac{1}{n} \sum_{k=1}^n \omega_k \right) \\ &= \frac{1}{2} \left(\sum_{j=1}^n \omega_j + \sum_{j=1}^n \omega_{n-j+2} \right) - \sum_{k=1}^n \omega_k \\ &= \frac{1}{2} \left(\sum_{j=1}^n \omega_j + \sum_{m=1}^n \omega_m \right) - \sum_{k=1}^n \omega_k = 0. \end{aligned}$$

From this equality, we derive

$$\sum_{j=2}^n \bar{\mu}_j = -\bar{\mu}_1.$$

Therefore, we get

$$\sum_{j=2}^n \bar{\mu}_j C_j \leq -\bar{\mu}_1 C_1,$$

so that

$$\bar{\mu}_1 C_1 + \sum_{j=2}^n \bar{\mu}_j C_j \leq 0.$$

It follows from this and Equation (8) that

$$\sum_{j=1}^n \omega_j C_j \leq \left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \sum_{j=1}^n C_j.$$

Due to the cyclic symmetry of $x_1, \dots, x_n$, and setting $m = i + j - 1$, we have

$$\begin{aligned} \sum_{j=1}^n C_j &= \sum_{j=1}^n \sum_{i=1}^n x_i x_{i+j-1} = \sum_{i=1}^n x_i \left(\sum_{j=1}^n x_{i+j-1} \right) \\ &= \sum_{i=1}^n x_i \left(\sum_{m=1}^n x_m \right) = \left(\sum_{i=1}^n x_i \right)^2, \end{aligned}$$

so that

$$\sum_{j=1}^n \omega_j C_j \leq \left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \left(\sum_{i=1}^n x_i \right)^2.$$

It follows from this and Equation (6) that

$$\sum_{i=1}^n x_i \sum_{j=1}^n \omega_j x_{i+j-1} \leq \left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \left(\sum_{i=1}^n x_i \right)^2.$$

Combining this with Equation (5) yields

$$\sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} \geq \frac{\left(\sum_{i=1}^n x_i^{\frac{p+2}{2}} \right)^2}{\left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \left(\sum_{i=1}^n x_i \right)^2}. \quad (9)$$

Let us now bound the last numerator.

Let $r = (p+2)/2$. Since $p \geq 0$, we have $r \geq 1$. The power mean inequality states that

$$\left(\frac{1}{n} \sum_{i=1}^n x_i^r \right)^{\frac{1}{r}} \geq \frac{1}{n} \sum_{i=1}^n x_i.$$

Isolating the sum and squaring both sides gives

$$\left(\sum_{i=1}^n x_i^r \right)^2 \geq n^{2-2r} \left(\sum_{i=1}^n x_i \right)^{2r},$$

so that

$$\left(\sum_{i=1}^n x_i^{\frac{p+2}{2}} \right)^2 \geq n^{-p} \left(\sum_{i=1}^n x_i \right)^{p+2}.$$

It follows from this and Equation (9) that

$$\begin{aligned} \sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} &\geq \frac{n^{-p} \left(\sum_{i=1}^n x_i \right)^{p+2}}{\left(\frac{1}{n} \sum_{k=1}^n \omega_k \right) \left(\sum_{i=1}^n x_i \right)^2} \\ &= \frac{n}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^p. \end{aligned}$$

The equality occurs if $x_1 = \dots = x_n$, with both sides equal to $(n/\sum_{k=1}^n \omega_k) x_1^p$. It also occurs if $p = 0$ and $\omega_1 = \dots = \omega_n$, with both sides equal to $1/\omega_1$. This completes the proof of the theorem. $\square$

Setting $p = 0$, and relabeling the variables as $(x_1, x_2, x_3) = (a, b, c)$ and the parameters as $(\omega_1, \omega_2, \omega_3) = (\alpha, \beta, \gamma)$, Theorem 3.1 reduces to Theorem 2.1. Indeed, it is sufficient to remark that the parameter condition in Equation (4) reduces to

$$\beta + \gamma \geq \frac{2}{3}(\alpha + \beta + \gamma),$$

which is equivalent to

$$2\alpha \leq \beta + \gamma.$$

We recover the parameter condition of Theorem 2.1. Furthermore, we observe that

$$\begin{aligned} & \frac{a}{\alpha a + \beta b + \gamma c} + \frac{b}{\alpha b + \beta c + \gamma a} + \frac{c}{\alpha c + \beta a + \gamma b} \\ &= \sum_{i=1}^n \frac{x_i}{\sum_{j=1}^n \omega_j x_{i+j-1}} \geq \frac{n}{\sum_{k=1}^n \omega_k} = \frac{3}{\alpha + \beta + \gamma}. \end{aligned}$$

Theorem 3.1 extends the inequality in Theorem 2.1 to the multivariable setting while incorporating an adjustable parameter p . It thereby complements [10, Theorem 7] in two key respects.

3.2 Applications

In this subsection, we present several applications of Theorem 3.1 in the form of corollaries, yielding new cyclic inequalities.

We first specialize in the four-variable case in the following corollary.

Corollary 3.4. *Let $p \geq 0$ and $\alpha, \beta, \gamma, \delta \geq 0$ be parameters such that $\alpha + \beta + \gamma + \delta > 0$,*

$$\beta + \delta \geq \alpha + \gamma, \quad 3\gamma \geq \alpha + \beta + \delta.$$

For any $a, b, c, d > 0$, the following cyclic inequality holds:

$$\begin{aligned} & \frac{a^{p+1}}{\alpha a + \beta b + \gamma c + \delta d} + \frac{b^{p+1}}{\alpha b + \beta c + \gamma d + \delta a} + \frac{c^{p+1}}{\alpha c + \beta d + \gamma a + \delta b} \\ &+ \frac{d^{p+1}}{\alpha d + \beta a + \gamma b + \delta c} \geq \frac{4}{\alpha + \beta + \gamma + \delta} \left(\frac{a + b + c + d}{4} \right)^p. \end{aligned}$$

The equality occurs if $a = b = c = d$, or if $p = 0$ and $\alpha = \beta = \gamma = \delta$.

Proof. We set $n = 4$ in Theorem 3.1 and relabel the variables as $(x_1, x_2, x_3, x_4) = (a, b, c, d)$ and the parameters as $(\omega_1, \omega_2, \omega_3, \omega_4) = (\alpha, \beta, \gamma, \delta)$. The parameter constraints arise from evaluating the parameter condition in Equation (4) for the valid indices $j = 2, 3, 4$. When $j = 2$, the index equation evaluates to

$$\omega_2 + \omega_4 \geq \frac{2}{4} \sum_{k=1}^n \omega_k.$$

Substituting the parameters gives

$$\beta + \delta \geq \frac{1}{2}(\alpha + \beta + \gamma + \delta),$$

which is equivalent to

$$\beta + \delta \geq \alpha + \gamma.$$

When $j = 3$, the index equation evaluates to

$$\omega_3 + \omega_3 \geq \frac{2}{4} \sum_{k=1}^n \omega_k.$$

Substituting the parameters gives

$$2\gamma \geq \frac{1}{2}(\alpha + \beta + \gamma + \delta),$$

which is equivalent to

$$3\gamma \geq \alpha + \beta + \delta.$$

For $j = 4$, the parameter condition coincides with the case $j = 2$.

Furthermore, we observe that

$$\begin{aligned} & \frac{a^{p+1}}{\alpha a + \beta b + \gamma c + \delta d} + \frac{b^{p+1}}{\alpha b + \beta c + \gamma d + \delta a} + \frac{c^{p+1}}{\alpha c + \beta d + \gamma a + \delta b} \\ & + \frac{d^{p+1}}{\alpha d + \beta a + \gamma b + \delta c} = \sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} \\ & \geq \frac{n}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^p = \frac{4}{\alpha + \beta + \gamma + \delta} \left(\frac{a+b+c+d}{4} \right)^p. \end{aligned}$$

The equality case follows from Theorem 3.1. This ends the proof of the corollary. $\square$

Setting $p = 0$, the right-hand side simplifies significantly, yielding a cyclic fractional inequality bounding the sum of the four ratios. Under identical parameter conditions, we obtain

$$\begin{aligned} & \frac{a}{\alpha a + \beta b + \gamma c + \delta d} + \frac{b}{\alpha b + \beta c + \gamma d + \delta a} + \frac{c}{\alpha c + \beta d + \gamma a + \delta b} \\ & + \frac{d}{\alpha d + \beta a + \gamma b + \delta c} \geq \frac{4}{\alpha + \beta + \gamma + \delta}. \end{aligned}$$

The five-variable case, which is likewise new, is presented in the following corollary.

Corollary 3.5. Let $p \geq 0$ and $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$ be parameters such that $\alpha + \beta + \gamma + \delta + \varepsilon > 0$,

$$3(\beta + \varepsilon) \geq 2(\alpha + \gamma + \delta), \quad 3(\gamma + \delta) \geq 2(\alpha + \beta + \varepsilon).$$

For any $a, b, c, d, e > 0$, the following cyclic inequality holds:

$$\begin{aligned} & \frac{a^{p+1}}{\alpha a + \beta b + \gamma c + \delta d + \varepsilon e} + \frac{b^{p+1}}{\alpha b + \beta c + \gamma d + \delta e + \varepsilon a} + \frac{c^{p+1}}{\alpha c + \beta d + \gamma e + \delta a + \varepsilon b} \\ & + \frac{d^{p+1}}{\alpha d + \beta e + \gamma a + \delta b + \varepsilon c} + \frac{e^{p+1}}{\alpha e + \beta a + \gamma b + \delta c + \varepsilon d} \\ & \geq \frac{5}{\alpha + \beta + \gamma + \delta + \varepsilon} \left(\frac{a+b+c+d+e}{5} \right)^p. \end{aligned}$$

The equality occurs if $a = b = c = d = e$, or if $p = 0$ and $\alpha = \beta = \gamma = \delta = \varepsilon$.

Proof. We set $n = 5$ in Theorem 3.1 and relabel the variables as $(x_1, x_2, x_3, x_4, x_5) = (a, b, c, d, e)$ and the parameters as $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = (\alpha, \beta, \gamma, \delta, \varepsilon)$. The parameter constraints arise from evaluating the parameter condition in Equation (4) for the valid indices $j = 2, 3, 4, 5$. When $j = 2$, the index equation evaluates to

$$\omega_2 + \omega_5 \geq \frac{2}{5} \sum_{k=1}^n \omega_k.$$

Substituting the parameters gives

$$\beta + \varepsilon \geq \frac{2}{5}(\alpha + \beta + \gamma + \delta + \varepsilon),$$

which is equivalent to

$$5(\beta + \varepsilon) \geq 2(\alpha + \beta + \gamma + \delta + \varepsilon),$$

so that

$$3(\beta + \varepsilon) \geq 2(\alpha + \gamma + \delta).$$

When $j = 3$, the index equation evaluates to

$$\omega_3 + \omega_4 \geq \frac{2}{5} \sum_{k=1}^n \omega_k.$$

Substituting the parameters gives

$$\gamma + \delta \geq \frac{2}{5}(\alpha + \beta + \gamma + \delta + \varepsilon),$$

which is equivalent to

$$5(\gamma + \delta) \geq 2(\alpha + \beta + \gamma + \delta + \varepsilon),$$

so that

$$3(\gamma + \delta) \geq 2(\alpha + \beta + \varepsilon).$$

For $j = 4$ and $j = 5$, the parameter conditions coincide with the cases $j = 3$ and $j = 2$, respectively.

Furthermore, we observe that

$$\begin{aligned} & \frac{a^{p+1}}{\alpha a + \beta b + \gamma c + \delta d + \varepsilon e} + \frac{b^{p+1}}{\alpha b + \beta c + \gamma d + \delta e + \varepsilon a} + \frac{c^{p+1}}{\alpha c + \beta d + \gamma e + \delta a + \varepsilon b} \\ & + \frac{d^{p+1}}{\alpha d + \beta e + \gamma a + \delta b + \varepsilon c} + \frac{e^{p+1}}{\alpha e + \beta a + \gamma b + \delta c + \varepsilon d} = \sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} \\ & \geq \frac{n}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^p = \frac{5}{\alpha + \beta + \gamma + \delta + \varepsilon} \left(\frac{a + b + c + d + e}{5} \right)^p. \end{aligned}$$

The equality case follows from Theorem 3.1. This concludes the proof of the corollary. $\square$

Setting $p = 0$, the right-hand side simplifies significantly, yielding a cyclic fractional inequality bounding the sum of the five ratios. Under identical parameter conditions, we obtain

$$\begin{aligned} & \frac{a}{\alpha a + \beta b + \gamma c + \delta d + \varepsilon e} + \frac{b}{\alpha b + \beta c + \gamma d + \delta e + \varepsilon a} + \frac{c}{\alpha c + \beta d + \gamma e + \delta a + \varepsilon b} \\ & + \frac{d}{\alpha d + \beta e + \gamma a + \delta b + \varepsilon c} + \frac{e}{\alpha e + \beta a + \gamma b + \delta c + \varepsilon d} \geq \frac{5}{\alpha + \beta + \gamma + \delta + \varepsilon}. \end{aligned}$$

In the following section, we extend Theorem 3.1 to a functional setting.

4 General functional framework

The functional generalization of Theorem 3.1 is presented in the following theorem.

Theorem 4.1. Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$, $p \geq 0$, $\omega_1, \dots, \omega_n \geq 0$ be parameters such that $\sum_{k=1}^n \omega_k > 0$ and, for any $j = 2, \dots, n$,

$$\omega_j + \omega_{n-j+2} \geq \frac{2}{n} \sum_{k=1}^n \omega_k,$$

and $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex and increasing function. For any $x_1, \dots, x_n > 0$ with the indices taken cyclically modulo n (i.e., $x_{n+k} = x_k$ for any $k = 1, \dots, n$), the following cyclic inequality holds:

$$\sum_{i=1}^n f \left(\frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} \right) \geq n f \left(\frac{1}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^p \right).$$

The equality occurs if $x_1 = \dots = x_n$, or $p = 0$ and $\omega_1 = \dots = \omega_n$.

Proof. The proof relies on applying the Jensen inequality for convex functions to f , which yields

$$\begin{aligned} \sum_{i=1}^n f\left(\frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}}\right) &= n \left(\frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}}\right) \right) \\ &\geq n f\left(\frac{1}{n} \sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}}\right). \end{aligned}$$

It follows from Theorem 3.1 and the fact that f is increasing that

$$\begin{aligned} f\left(\frac{1}{n} \sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}}\right) &\geq f\left(\frac{1}{n} \times \frac{n}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i\right)^p\right) \\ &= f\left(\frac{1}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i\right)^p\right). \end{aligned}$$

Therefore, we derive

$$\sum_{i=1}^n f\left(\frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}}\right) \geq n f\left(\frac{1}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i\right)^p\right).$$

The equality occurs if $x_1 = \dots = x_n$, with both sides equal to $n f(x_1^p / \sum_{k=1}^n \omega_k)$. It also occurs if $p = 0$ and $\omega_1 = \dots = \omega_n$, with both sides equal to $n f(1/(n\omega_1))$. This completes the proof of the theorem. $\square$

We now state several immediate corollaries of Theorem 4.1 for various convex and increasing functions f , starting with the power function case.

Corollary 4.2. Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$, $q \geq 1$, and $\omega_1, \dots, \omega_n \geq 0$ be parameters such that $\sum_{k=1}^n \omega_k > 0$ and, for any $j = 2, \dots, n$,

$$\omega_j + \omega_{n-j+2} \geq \frac{2}{n} \sum_{k=1}^n \omega_k.$$

For any $x_1, \dots, x_n > 0$ with the indices taken cyclically modulo n (i.e., $x_{n+k} = x_k$ for any $k = 1, \dots, n$), the following cyclic inequality holds:

$$\sum_{i=1}^n \left(\frac{x_i}{\sum_{j=1}^n \omega_j x_{i+j-1}} \right)^q \geq n \left(\frac{1}{\sum_{k=1}^n \omega_k} \right)^q.$$

The equality occurs if $x_1 = \dots = x_n$, or if $\omega_1 = \dots = \omega_n$.

Proof. The proof is a direct consequence of Theorem 4.1 by setting $p = 0$ and considering the function $f(t) = t^q$ for $q \geq 1$. Note that f is both increasing and convex on $(0, \infty)$, as verified by its derivatives $f'(t) = qt^{q-1} > 0$ and $f''(t) = q(q-1)t^{q-2} \geq 0$. This ends the proof of the corollary. $\square$

In particular, setting $q = 2$ in the framework of this corollary, we obtain

$$\begin{aligned} \sum_{i=1}^n \frac{x_i^2}{\left(\sum_{j=1}^n \omega_j x_{i+j-1}\right)^2} &= \sum_{i=1}^n \left(\frac{x_i}{\sum_{j=1}^n \omega_j x_{i+j-1}} \right)^2 \geq n \left(\frac{1}{\sum_{k=1}^n \omega_k} \right)^2 \\ &= \frac{n}{\left(\sum_{k=1}^n \omega_k\right)^2}. \end{aligned}$$

The following corollary applies Theorem 4.1 to a convex and increasing hyperbolic function while incorporating a general parameter $p \geq 0$.

Corollary 4.3. Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$, $p \geq 0$, and $\omega_1, \dots, \omega_n \geq 0$ be parameters such that $\sum_{k=1}^n \omega_k > 0$ and, for any $j = 2, \dots, n$,

$$\omega_j + \omega_{n-j+2} \geq \frac{2}{n} \sum_{k=1}^n \omega_k.$$

For any $x_1, \dots, x_n > 0$ with the indices taken cyclically modulo n (i.e., $x_{n+k} = x_k$ for any $k = 1, \dots, n$), the following cyclic inequality holds:

$$\sum_{i=1}^n \sinh \left(\frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} \right) \geq n \sinh \left(\frac{1}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^p \right).$$

The equality occurs if $x_1 = \dots = x_n$, or if $p = 0$ and $\omega_1 = \dots = \omega_n$.

Proof. The proof follows from Theorem 4.1 by applying it to the function $f(t) = \sinh(t)$, which is both increasing and convex on $(0, \infty)$ since $f'(t) = \cosh(t) > 0$ and $f''(t) = \sinh(t) > 0$. This concludes the proof of the corollary. $\square$

The following corollary applies Theorem 4.1 to a (polynomial-)exponential function.

Corollary 4.4. Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$, $\lambda > 0$, and $\omega_1, \dots, \omega_n \geq 0$ be parameters such that $\sum_{k=1}^n \omega_k > 0$ and, for any $j = 2, \dots, n$,

$$\omega_j + \omega_{n-j+2} \geq \frac{2}{n} \sum_{k=1}^n \omega_k.$$

For any $x_1, \dots, x_n > 0$ with the indices taken cyclically modulo n (i.e., $x_{n+k} = x_k$ for any $k = 1, \dots, n$), the following cyclic inequality holds:

$$\sum_{i=1}^n \frac{x_i}{\sum_{j=1}^n \omega_j x_{i+j-1}} \exp \left(\frac{\lambda x_i}{\sum_{j=1}^n \omega_j x_{i+j-1}} \right) \geq \frac{n}{\sum_{k=1}^n \omega_k} \exp \left(\frac{\lambda}{\sum_{k=1}^n \omega_k} \right).$$

The equality occurs if $x_1 = \dots = x_n$, or if $\omega_1 = \dots = \omega_n$.

Proof. The proof follows from Theorem 4.1 by setting $p = 0$ and applying it to the function $f(t) = te^{\lambda t}$ for $\lambda > 0$, which is both increasing and convex on $(0, \infty)$ since $f'(t) = (\lambda t + 1)e^{\lambda t} > 0$ and $f''(t) = \lambda(\lambda t + 2)e^{\lambda t} > 0$. This completes the proof of the corollary. $\square$

These corollaries illustrate the versatility of Theorem 4.1 in generating new functional cyclic inequalities.

5 Conclusion

In this paper, we have proposed a comprehensive framework for parametric cyclic inequalities, extending classical Nesbitt-type bounds into a generalized multivariable and functional setting. We began by reviewing and proposing an alternative proof to [10, Theorem 7]. This first result was subsequently extended to an n -variable framework with the introduction of arbitrary parameters $\omega_1, \dots, \omega_n \geq 0$ and a structural parameter $p \geq 0$. Through the symmetrization of quadratic forms and the application of the power mean inequality, we derived the general parametric constraint, for any $j = 2, \dots, n$,

$$\omega_j + \omega_{n-j+2} \geq \frac{2}{n} \sum_{k=1}^n \omega_k.$$

The last contribution of this work is Theorem 4.1, which extends these algebraic inequalities to a general functional framework. By using the properties of convex and increasing functions, we unify a broad class of cyclic bounds, via power, exponential, and hyperbolic forms, under a single result.

6 Open Problem

While Theorem 2.1 (and its multivariable generalization, Theorem 3.1) establishes that $2\alpha \leq \beta + \gamma$, and more generally, the multivariable condition $\omega_j + \omega_{n-j+2} \geq (2/n) \sum_{k=1}^n \omega_k$ for any $j = 2, \dots, n$, is sufficient for the inequality to hold, it does not provide a full characterization of the admissible parameter space. Indeed, as observed in the three-variable case with $\alpha = 1$ and $\beta = \gamma = 0$, the inequality holds despite violating $2\alpha \leq \beta + \gamma$.

Open Problem 6.1. Determine the necessary and sufficient conditions on the parameters $\omega_1, \dots, \omega_n$ (or α, β, γ when $n = 3$) that precisely delineate the full admissible domain for which the cyclic inequality

$$\sum_{i=1}^n \frac{x_i^{p+1}}{\sum_{j=1}^n \omega_j x_{i+j-1}} \geq \frac{n}{\sum_{k=1}^n \omega_k} \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^p$$

holds for any $x_1, \dots, x_n > 0$ and $p \geq 0$.

Declarations

Funding. The author received no external funding for this research.

Conflicts of interest. The author declares that he has no competing interests.

Ethics approval. Not applicable.

Informed consent. Not applicable.

Data availability. No data were generated or analysed during this study.

Author contributions. The author is solely responsible for the conceptualization, analysis, and writing of this manuscript.

Use of artificial intelligence. No AI tools were used beyond language editing.

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